This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

1824 e. 1

600050037L

~~50.406.~~

1824 e. 1

CORRIGENDA.

Page 12. In the second line in the proof of Herschel's theorem, insert

$$\therefore f(e^t) = \&\text{c.}$$

Page 16. The Ex. in Art. 32, will not be correct, except *5* and not 1 be taken for the increment.

Page 60. The complementary function may be further simplified into

$$a^x C_{y+x} + \dots,$$
$$+ b^x ({}_0C_{y+x} + {}_1C_{y+x}x + \dots + {}_{p-1}C_{y+x}x^{p-1}).$$

troduced this variable, and performed the operation, regarding unity for its increment, we can express the result in terms of x by writing $\frac{x}{h}$ for z.

Differences of Explicit Functions.

3. To find the difference of $Au_x + B$; u_x being a function of x, and A and B quantities which do not depend on x.

$$\Delta(Au_x + B) = (Au_{x+1} + B) - (Au_x + B) = Au_{x+1} - Au_x = A\Delta u_x.$$

Hence when $A = 0$, we have $\Delta B = 0$.

4. To find the difference of a number of functions of x, connected by the operations of addition or subtraction.

$$\begin{aligned}\Delta(u_x + v_x - w_x) &= (u_{x+1} + v_{x+1} - w_{x+1}) - (u_x + v_x - w_x)\\ &= (u_{x+1} - u_x) + (v_{x+1} - v_x) - (w_{x+1} - w_x)\\ &= \Delta u_x + \Delta v_x - \Delta w_x.\end{aligned}$$

5. To find the difference of the product of two functions.

$$\begin{aligned}\Delta u_x v_x &= u_{x+1}v_{x+1} - u_x v_x = u_{x+1}v_{x+1} - u_x v_{x+1} + u_x v_{x+1} - u_x v_x\\ &= v_{x+1}(u_{x+1} - u_x) + u_x(v_{x+1} - v_x)\\ &= v_{x+1}\Delta u_x + u_x \Delta v_x.\end{aligned}$$

6. To find the difference of the quotient of two functions.

$$\begin{aligned}\Delta \frac{u_x}{v_x} &= \frac{u_{x+1}}{v_{x+1}} - \frac{u_x}{v_x} = \frac{u_{x+1}v_x - v_{x+1}u_x}{v_{x+1}v_x},\\ &= \frac{(u_x + \Delta u_x)v_x - (v_x + \Delta v_x)u_x}{v_{x+1}v_x}\\ &= \frac{v_x\Delta u_x - u_x\Delta v_x}{v_{x+1}v_x}.\end{aligned}$$

7. To find the difference of the continued product of any number of successive values of a function.

Let $u_x u_{x+1} \dots\dots u_{x+n}$ be the product.

$$\Delta(u_x u_{x+1} \dots\dots u_{x+n}) = u_{x+1} \dots\dots u_{x+n+1} - u_x u_{x+1} \dots\dots u_{x+n};$$
$$= (u_{x+1} \dots\dots u_{x+n})(u_{x+n+1} - u_x).$$

Hence if $u_x = A_0 + A_1 x$,

$$\Delta u_x u_{x+1} \dots\dots u_{x+n} = (n+1)A_1(u_{x+1} u_{x+2} \dots\dots u_{x+n}).$$

8. To find the difference of the quotient of continued products of any number of successive values of two functions u_x and v_x.

Let $\dfrac{u_x u_{x+1} \dots\dots u_{x+n}}{v_x v_{x+1} \dots\dots v_{x+m}}$, be the quotient,

$$\Delta \frac{u_x u_{x+1} \dots\dots u_{x+n}}{v_x v_{x+1} \dots\dots v_{x+m}} = \frac{u_{x+1} u_{x+2} \dots\dots u_{x+n+1}}{v_{x+1} v_{x+2} \dots\dots v_{x+m+1}} - \frac{u_x u_{x+1} \dots\dots u_{x+n}}{v_x v_{x+1} \dots\dots v_{x+m}},$$
$$= \frac{u_{x+1} u_{x+2} \dots\dots u_{x+n}}{v_{x+1} v_{x+2} \dots\dots v_{x+m}} \left(\frac{u_{x+n+1}}{v_{x+m+1}} - \frac{u_x}{v_x}\right);$$
$$= \frac{u_{x+1} \dots\dots u_{x+n}}{v_x v_{x+1} \dots\dots v_{x+m+1}} (v_x u_{x+n+1} - u_x v_{x+m+1}).$$

Hence where the numerator is unity, and $v_x = A_0 + A_1 x$,

$$\Delta \frac{1}{v_x v_{x+1} \dots\dots v_{x+m}} = - \frac{(m+1)A_1}{v_x v_{x+1} \dots\dots v_{x+m+1}},$$

9. To find the differences of any rational integral function.

Let $ax^n + bx^{n-1} + \dots\dots kx + l = u_x$ be the function. Then

$$\Delta u_x = \{a(x+1)^n + b(x+1)^{n-1} + \dots\dots k(x+1) + l\},$$
$$- (ax^n + bx^{n-1} + \dots\dots + kx + l),$$
$$= nax^{n-1} + b_1 x^{n-2} + \dots\dots i_1 x + k_1,$$

which is a rational integral function lower by one dimension than that from which it is derived.

Repeating the operation, and denoting the result or $\Delta(\Delta u_x)$ by $\Delta^2 u_x$, we have

$$\Delta^2 u_x = n.(n-1)\,ax^{n-2} + b_2 x^{n-3} + \ldots\ldots h_2 x + i_2,$$

and so on, and finally, when we have repeated the operation (n) times, using the same notation,

$$\Delta^n u_x = n.(n-1)\ldots\ldots 3.2.1a.$$

Hence the n^{th} difference is constant, and therefore all differences of higher orders vanish.

10. To find the differences of $\sin u_x$ and $\cos u_x$.

$$\begin{aligned}\Delta \sin u_x &= \sin u_{x+1} - \sin u_x = \sin(u_x + \Delta u_x) - \sin u_x,\\ &= 2 \sin \frac{\Delta u_x}{2} \cos\left(u_x + \frac{\Delta u_x}{2}\right),\\ &= 2 \sin \frac{\Delta u_x}{2} \sin \left\{u_x + \tfrac{1}{2}(\pi + \Delta u_x)\right\},\end{aligned}$$

$$\begin{aligned}\Delta \cos u_x &= \cos u_{x+1} - \cos u_x = \cos(u_x + \Delta u_x) - \cos u_x,\\ &= -\,2 \sin \frac{\Delta u_x}{2} \sin\left(u_x + \frac{\Delta u_x}{2}\right),\\ &= 2 \sin \frac{\Delta u_x}{2} \cos \left\{u_x + \tfrac{1}{2}(\pi + \Delta u_x)\right\}.\end{aligned}$$

Making $u_x = ax + \beta$, and repeating the operation (n) times,

$$\Delta^n \sin(ax+\beta) = \left(2 \sin \frac{a}{2}\right)^n \sin \left\{(ax+\beta) + \frac{n}{2}(\pi + a)\right\},$$

$$\Delta^n \cos(ax+\beta) = \left(2 \sin \frac{a}{2}\right)^n \cos \left\{(ax+\beta) + \frac{n}{2}(\pi + a)\right\}.$$

11. To find the differences of $\tan u_x$, and $\tan^{-1} u_x$.

$$\Delta \tan u_x = \tan (u_x + \Delta u_x) - \tan u_x,$$

$$= \frac{\sin (u_x + \Delta u_x) \cos u_x - \cos (u_x + \Delta u_x) \sin u_x}{\cos u_{x+1} \cos u_x},$$

$$= \frac{\sin \Delta u_x}{\cos u_{x+1} \cos u_x}.$$

Whence, $\Delta \tan ax = \dfrac{\sin a}{\cos (x + 1) a \cos ax}$,

$$\Delta \tan^{-1} u_x = \tan^{-1} u_{x+1} - \tan^{-1} u_x,$$

$$= \tan^{-1} \frac{u_{x+1} - u_x}{1 + u_{x+1} u_x} = \tan^{-1} \frac{\Delta u_x}{1 + u_{x+1} u_x}.$$

Whence, $\Delta \tan^{-1} ax = \tan^{-1} \dfrac{a}{1 + (x + 1) x a^2}$.

12. To find the differences of a^x,

$$\Delta a^x = a^{x+1} - a^x = (a - 1) a^x.$$

Repeating the operation,

$$\Delta^2 a^x = (a - 1)^2 a^x,$$

and so on, till we come to

$$\Delta^n u_x = (a - 1)^n a^x.$$

13. To find the difference of $\log u_x$.

$$\Delta \log u_x = \log u_{x+1} - \log u_x = \log \frac{u_{x+1}}{u_x},$$

$$= \log \left(1 + \frac{\Delta u_x}{u_x} \right),$$

$$= \frac{\Delta u_x}{u_x} - \frac{1}{2} \left(\frac{\Delta u_x}{u_x} \right)^2 + \frac{1}{3} \left(\frac{\Delta u_x}{u_x} \right)^3 \ldots . \ \&c.$$

Successive Values of a Function.

14. If in any function of x, as u_x, we substitute successively the values $x + h$, $x + 2h$ .... $x + nh$, for x, the results are termed the successive values of the function.

The first successive value is thus expressed, $Du_x = u_{x+h}$; and h is generally unity, except where the contrary is stated.

Hence, $D^2u_x = Du_{x+h} = u_{x+2h}$, $D^3u_x = u_{x+3h}$, &c.,

and generally, $D^nu_x = u_{x+nh}$.

Relations between the Successive Values and the Differences of a Function.

15. These relations we propose to investigate on what is termed the method of separation of symbols of operation and quantity, before applying which it will be necessary to make a few remarks by way of explanation.

By a symbol of operation is meant a symbol put for conciseness to denote that a defined operation is performed on the function, whether general or specified, to which it refers. Thus the symbol Δ used above denotes that two different values of the variable are substituted in the function and the results subtracted from each other. In some cases the function is expressed, as where we write Δu_x; in others we separate the symbols Δ and u_x. Thus, in place of

$$D^nu_x + p_1D^{n-1}u_x + \dots + p_nu_x,$$

we write $$(D^n + p_1D^{n-1} + \dots + p_n)\, u_x.$$

The legitimacy of this step appears from considering the relation in which the symbol u_x stands to the symbols within the brackets. The symbol u_x is merely a representative of functions of x in general, just as in algebra we denote by a single symbol numbers in general: and in the expression Du_x the symbol u_x appears in this character, namely as an affix, a function for the symbol D to operate upon. Hence it may be altogether effaced, provided we keep in mind that it (or some other representative of functions) is that to which the symbol D refers; just as in algebra we omit the unit to which a symbol, as (a), refers. Therefore also it may be separated from the D, as in the expression

$$\left(D^n + nD^{n-1} + \frac{n(n-1)}{1.2} D^{n-2} + \dots\right) u_x,$$

and this notation is rendered consistent from the analogous modes of expression according to which we place the u_x after the D in Du_x, to denote that the D operates on the u_x, and the u_x after the symbol of quantity a in au_x to denote that a multiplies the u_x.

By a Symbol of Quantity is meant a symbol put for a number or complicated algebraical expression.

Symbols of quantity may be regarded, however, as symbols of operation. Thus if a be a number, it denotes that unity, whether the unit be abstract or concrete, is to be repeated or added to itself (a) times, and a^2 or $a.a$ that the same operation is to be performed on a as was on unity, and a^3 or $a.a^2$ in like manner:* and in all cases where the two kinds of symbols occur together, as for example in $a + \Delta$, we must regard a as a symbol of operation for the sake of consistency.

16. In the proofs on which are founded a variety of theorems in ordinary algebra, the only properties required to characterise the symbols employed are, that they are subject to three laws, denominated by a late writer, the commutative, distributive, and index laws.

(1) The commutative, $ab = ba$,

(2) The distributive, $c(a + b) = ca + cb$.

(3) The index law, $a^m.(a^n) = a^{m+n}$.

---

* If in place of the expression, raising of powers, which involves the notion of multiplication, we use as a corresponding term one implying that the operation is performed successively, it would have this convenience that, we could thus interpret the meaning of the m in such symbols as a^m, Δ^m, D^m, $\left(\frac{d}{dx}\right)^m$ in the same words.

Hence these proofs hold for all symbols which are subject to these laws, whatever be their nature in other respects. Now if the symbols which are used in this calculus, Δ and D, (the functions they are to operate upon being *understood*) are so subject, then such theorems as those above referred to will hold for these symbols. Thus, from the theorem

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m.(m-1)}{1.2}a^{m-2}b^2 + \ldots.$$

we have also

$$(\Delta + D)^m = \Delta^m + m\Delta^{m-1}D + \frac{m.(m-1)}{1.2}\Delta^{m-2}D^2 + \ldots.$$

17. Now that the symbols D and Δ follow: (1) the commutative, (2) the distributive, and (3) the index laws, may be proved as follows. For the symbol D,

(1) If $u_{x,y}$ be a function of x and y, and D refer to x, and D' to y only, we have

$$D'u_{x,y} = u_{x,y+k},$$

$$\therefore\ DD'u_{x,y} = Du_{x,y+k} = u_{x+h,y+k} = D'u_{x+h,y} = D'Du_{x,y},$$

$$\therefore\ DD' = D'D.$$

(2) If u_x, v_x be functions of x,

$$Du_x + Dv_x = u_{x+h} + v_{x+h} = D(u_x + v_x).$$

(3) As we have before shewn,

$$D^n u_x = u_{x+nh},$$

$$\therefore\ D^m D^n u_x = Du_{x+nh} = u_{x+nh+mh} = D^{m+n}u_x,$$

$$\therefore\ D^m D^n = D^{m+n}.$$

18. For the symbol Δ,

(1) If $u_{x,y}$ denote a function of x and y, and Δ refer to x, and Δ' to y, only,

$$\Delta' u_{x,y} = u_{x,y+k} - u_{x,y},$$

$$\therefore \ \Delta\Delta' u_{x,y} = \Delta u_{x,y+k} - \Delta u_{x,y} = u_{x+h,y+k} - u_{x,y+k} - (u_{x+h,y} - u_{x,y})$$

$$= u_{x+h,y+k} - u_{x+h,y} - (u_{x,y+k} - u_{x,y}) = \Delta' u_{x+h,y} - \Delta' u_{x,y} = \Delta'\Delta u_{x,y},$$

$$\therefore \ \Delta\Delta' = \Delta'\Delta.$$

(2) If u_x, v_x be functions of x,

$$\Delta u_x + \Delta v_x = u_{x+1} + v_{x+1} - (u_x + v_x) = \Delta(u_x + v_x).$$

(3) $\Delta^n u_x = (D - 1)^n u_x$,

$\therefore \ \Delta^m \Delta^n u_x = (D-1)^m (D-1)^n u_x = (D-1)^{m+n} u_x$ by the proof for D

$= \Delta^{m+n} u_x$,

$\therefore \ \Delta^m \Delta^n = \Delta^{m+n}$.

19. It might be easily shewn, further, that the symbols $\frac{d}{dx}$, D, Δ, referring to the same function, may be combined in any manner conformably with the above laws. Thus, for instance,

$$\frac{d}{dx}\Delta D = \frac{d}{dx} D\Delta = D\frac{d}{dx}\Delta = \text{\&c.},$$

and
$$\frac{d}{dx}(D + \Delta) = \frac{d}{dx} D + \frac{d}{dx}\Delta.$$

20. The definition of D is that $Du_x = u_{x+h}$, h being generally unity.

Now we know, by Taylor's theorem, that

$$u_{x+h} = e^{h\frac{d}{dx}} u_x,$$

where $e^{h\frac{d}{dx}}$ is a symbol of operation referring to u_x. Hence

$$D = e^{h\frac{d}{dx}}.$$

Also,
$$\Delta u_x = u_{x+h} - u_x = Du_x - u_x = (D - 1)\, u_x,$$

$$\therefore \ \Delta = D - 1.$$

21. To express $\Delta^n u_x$ by u_x and its n successive values.

Since
$$\Delta = D - 1,$$

$$\therefore\ \Delta^n = (D-1)^n = D^n - n.D^{n-1} + \frac{n.(n-1)}{1.2} D^{n-2} -, \text{\&c.},$$

or introducing a function u_x, since $D^n u_x = u_{x+nh}$,

$$\Delta^n u_x = u_{x+nh} - n.u_{x+(n-1)h} + \frac{n.(n-1)}{1.2} u_{x+(n-2)h} -, \text{\&c.}$$

22. If in this series we give to u_x a particular form, we easily obtain its n^{th} difference; thus let $u_x = x^m$,

$$\Delta^n\,(x^m) = (x+n)^m - n\,(x+n-1)^m + \frac{n.(n-1)}{1.2}(x+n-2)^m - \text{\&c.}$$

In this equation, if $n > m$, $\Delta^n\,(x^m) = 0$, whatever be the value of x. If $m = n$, $\Delta^n\,(x^m) = 1.2.3.\ldots.n$, and therefore

$$1.2.3\ldots n = (x+n)^n - n(x+n-1)^n + \frac{n.(n-1)}{1.2}(x+n-2)^n, - \text{\&c.}$$

and if $x = 0$, since x may have any value,

$$1.2\ldots n = n^n - n\,(n-1)^n + \frac{n.\,(n-1)}{1.2}(n-2)^n - \text{\&c.}$$

Also, making $x = 0$, we have

$$\Delta^n(0^m) = n^m - n.(n-1)^m + \frac{n.(n-1)}{1.2}(n-2)^m - \text{\&c.}$$

The numbers represented in the form $\Delta^n.0^m$, are much used in analytical calculations; they are termed the differences of zero, and afford a concise mode of expressing complicated formulæ.

We should find from this formula, making $n = 1, 2 \ldots . n$, successively,

$$\Delta 0^m = 1,$$

$$\Delta^2 0^2 = 2,\quad \Delta^2 0^3 = 6,\quad \Delta^2 0^4 = 14.\ldots$$

$$\Delta^3 0^3 = 6,\quad \Delta^3 0^4 = 36,\quad \Delta^3 0^5 = 150.\ldots$$

where $n > m$, the value is zero in all cases.

23. To express u_{x+nh} by u_x, and its first n differences.

Since $D = 1 + \Delta$,

$$D^n = (1 + \Delta)^n = 1 + n\Delta + \frac{n.(n-1)}{1.2}\Delta^2 + \frac{n.(n-1).(n-2)}{1.2.3}\Delta^3 ... + \Delta^n.$$

Introducing a function u_x, since $D^n u_x = u_{x+nh}$

$$u_{x+nh} = u_x + n\Delta u_x + \frac{n.(n-1)}{1.2}\Delta^2 u_x + ... + n\Delta^{n-1}u_x + \Delta^n u_x.$$

If $u_x = x^m$, $u_{x+n} = (x+n)^m$, h being unity,

$$\therefore (x+n)^m = (1+\Delta)^n x^m, \text{ or if } x = 0,$$

$$n^m = (1+\Delta)^n 0^m.$$

a formula by which any number is expressed in terms of the differences of zero.

24. Since $\Delta = D - 1$ and $D = e^{\frac{d}{dx}}$, h being unity,

$$\Delta^n = (e^{\frac{d}{dx}} - 1)^n,$$

and introducing the function

$$\Delta^n u_x = (e^{\frac{d}{dx}} - 1)^n u_x,$$

which is Lagrange's theorem.

25. To find the n^{th} difference of the product of two functions.

$$\Delta u_x v_x = u_{x+1} v_{x+1} - u_x v_x = Du_x Dv_x - u_x v_x = (DD' - 1)\, u_x v_x,$$

where D refers to u_x, and D' to v_x, only. Hence, repeating the operation n times,

$$\begin{aligned}\Delta^n u_x v_x &= (DD' - 1)^n u_x v_x \\ &= \left\{D^n D'^n - n.D^{n-1}D'^{n-1} + \frac{n.(n-1)}{1.2}D^{n-2}D'^{n-2} -..\right\} u_x v_x \\ &= u_{x+n} v_{x+n} - n.u_{x+n-1} v_{x+n-1} + \frac{n.(n-1)}{1.2} u_{x+n-2} v_{x+n-2} - \&c.\end{aligned}$$

We may expand in a different manner as follows. Since

$$\Delta u_x v_x = (DD' - 1)\,u_x v_x = \{D(1 + \Delta') - 1\}\,u_x v_x = (\Delta + D\Delta')\,u_x v_x$$

$$\therefore\ \Delta^n u_x v_x = (\Delta + D\Delta')^n u_x v_x$$

$$= \left\{\Delta^n + n.\Delta^{n-1}.D.\Delta' + \frac{n.(n-1)}{1.2}\Delta^{n-2}D^2.\Delta'^2 + \,.\,.\right\} u_x v_x$$

$$= v_x\Delta^n u_x + n\Delta v_x\Delta^{n-1}u_{x+1} + \frac{n.(n-1)}{1.2}\Delta^2 v_x\Delta^{n-2}u_{x+2} + \,.\,.\,.\,.$$

26. To prove Herschel's theorem for expanding any function of e^t, in a series of ascending powers of t.

Let $f(e^t) = A + Be^t + + Ce^{2t} + \,.\,.\,.\,.$

$$= (Ae^{xt} + Be^{(x+1)t} + Ce^{(x+2)t} + \,.\,.)_{(x=0)}$$

$$= (Ae^{xt} + Be^{\frac{d}{dx}}.\,e^{xt} + Ce^{2\frac{d}{dx}}e^{xt} + \,.\,.)_{(x=0)},$$

$$= (A + Be^{\frac{d}{dx}} + Ce^{2\frac{d}{dx}} + \,.\,.)\,e^{xt}._{(x=0)},$$

$$= f(e^{\frac{d}{dx}})\,.\,e^{xt}{}_{(x=0)},$$

$$= f(1 + \Delta)\,.\,e^{xt}._{(x=0)},$$

$$= f(1 + \Delta)\left\{1 + \frac{xt}{1} + \frac{x^2t^2}{1.2} + \,.\,.\right\}_{(x=0)}.$$

Hence the coefficient of t^m is

$$f(1 + \Delta).\,\frac{x^m}{1.2..m_{(x=0)}} \quad \text{or } f(1 + \Delta)\,\frac{0^m}{1.2...m},$$

which is Herschel's theorem.

Cor. Hence the general term in $(e^{\frac{d}{dx}} - 1)^n\,u_x$ is

$$\frac{\Delta^n 0^m}{1.2...m}\,\frac{d^m u_x}{dx^m}.$$

and therefore

$$\Delta^n u_x = \frac{d^n u_x}{dx^n} + \frac{\Delta^n 0^{n+1}}{1.2...(n+1)}\,\frac{d^{n+1}u_x}{dx^{n+1}} + \frac{\Delta^n 0^{n+2}}{1.2...(n+2)}\,\frac{d^{n+2}u_x}{dx^{n+2}} + \text{\&c.}$$

which is the general expression for the n^{th} difference of a function in terms of its differential coefficients.

27. To express the n^{th} differential coefficient of a function in terms of its differences.

Since $$e^{\frac{d}{dx}} = D = 1 + \Delta,$$

$$\frac{d}{dx} = \log(1+\Delta),$$

$$\therefore \frac{d^n}{dx^n} = \{\log(1+\Delta)\}^n;$$

or, introducing a function for the symbols to operate upon,

$$\frac{d^n u_x}{dx^n} = \{\log(1+\Delta)\}^n u_x.$$

This being true for all values of n, writing $-n$ for n, and remembering that $\left(\frac{d}{dx}\right)^{-n}$ is equivalent to $\int^n dx^n$,

$$\int^n dx^n u_x = \{\log(1+\Delta)\}^{-n} u_x.$$

CHAPTER II.

INVERSE METHOD OF DIFFERENCES.

28. THE Inverse Method of Differences is applied to determine from a given difference the function from which it was derived. This process is called integration.

The simplest case is where from the equation

$$\Delta u_x = f(x)$$

we have to determine u_x in terms of x. The result is thus expressed,

$$u_x = \Sigma f(x).$$

So that Σ and Δ denote operations the reverse of each other.

Another case is where we have to determine the form of u_x from the equation

$$\Delta^n u_x = f(x).$$

In this case, the result is written

$$u_x = \Sigma^n f(x).$$

29. Since any constant quantity connected with the variable part of an expression by the signs + or − disappears in taking the difference, it must be restored in integrating; and since all functions which differ from each other only in the values of the constant parts have the same difference, we must, in order to include all functions from which the proposed difference can have been derived, add to the integral an arbitrary con-

stant C. Instead of the constant, however, we may add any function of the variable x, which remains unchanged when $x+h$ is written for x. This remark will be found of importance.

30. Since $\Sigma^n\Delta^n u_x = u_x$ and according to the index law, whatever n be,

$$\Delta^{-n}\Delta^n u_x = \Delta^0 u_x = u_x;$$

Σ^n is identical with Δ^{-n}, and may be interchanged with it.

31. To find the integral of any rational integral function.

Since the difference of any rational integral function,

$$A_n x^n + A_{n-1}x^{n-1} + \ldots . + A_0 = f(x),$$

is a function of the same kind one dimension lower, it appears that, conversely, the integral of a function of this description is a similar function one dimension higher; hence, we may assume the integral of the above expression to be of the form,

$$p_{n+1}x^{n+1} + p_n x^n + \ldots . + p_0 = \Sigma f(x),$$

and then on taking the difference of each side, and equating the coefficients of like powers of x, we shall have $(n+1)$ simple equations by which to determine the $(n+1)$ quantities $p_1 p_2 \ldots . p_{n+1}$; the last term cannot be determined: it is the arbitrary constant which is required to give the integral its utmost generality.

Ex. Find $\Sigma(x^3 + 5x^2 + 6x)$.

Let $\Sigma(x^3 + 5x^2 + 6x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4$,

therefore taking the difference of each side,

$$x^3 + 5x^2 + 6x = p_1 + p_2(2x+1) + p_3(3x^2 + 3x + 1),$$
$$+ p_4(4x^3 + 6x^2 + 4x + 1)$$
$$= p_1 + p_2 + p_3 + p_4 + (2p_2 + 3p_3 + 4p_4)x + (3p_3 + 6p_4)x^2 + 4p_4 x^3,$$

equating coefficients of like powers,

$$4p_4 = 1, \quad \therefore\ p_4 = \tfrac{1}{4},$$

$$3p_3 + 6p_4 = 5, \qquad 3p_3 = 5 - \tfrac{3}{2} = \tfrac{7}{2}, \quad \therefore\ p_3 = \tfrac{7}{6},$$

$$2p_2 + 3p_3 + 4p_4 = 6, \qquad 2p_2 = 6 - 1 - \tfrac{7}{2} = \tfrac{3}{2}, \quad \therefore\ p_2 = \tfrac{3}{4}.$$

$$p_1 + p_2 + p_3 + p_4 = 0, \quad \therefore\ p_1 + \tfrac{13}{6} = 0\,; \quad \therefore\ p_1 = -\tfrac{13}{6}\,;$$

$$\therefore\ \Sigma(x^3 + 5x^2 + 6x) = \frac{x^4}{4} + \frac{7x^3}{6} + \frac{3x^2}{4} - \tfrac{13}{6}x + C.$$

32. Since if $u_x = A_0 + A_1 x$,

$$\Delta u_{x-1}u_x\,.\,u_{x+1}\ \ldots\ldots\ u_{x+n-1} = u_x\,.\,u_{x+1}\ \ldots\ldots\ u_{x+n-1}(u_{x+n} - u_{x-1})$$

$$= u_x u_{x+1}\ \ldots\ldots\ u_{x+n-1}(n+1)A_1.$$

Therefore taking the integral of both sides,

$$\Sigma u_x\ \ldots\ldots\ u_{x+n-1} = \frac{u_{x-1}u_x\ \ldots\ldots\ u_{x+n-1}}{(n+1)A_1} + C;$$

whence we deduce a rule for integrating the product of consecutive terms of an Arithmetic Progression, "Annex one more factor at the beginning, divide by the number of factors so increased, and by the common difference."

Ex. $\Sigma(x-10)(x-5)x.(x+5) = \dfrac{(x-15)(x-10)(x-5)x.(x+5)}{5.5} + C.$

It is convenient to resolve by inspection a rational integral function into factors as above, in place of integrating by the method in Art. 31.

In integrating by this rule, we must be careful to observe that each factor must be derivable from the preceding by writing $x + 1$ for x, thus the expression

$$(2x - 1)\,.\,2x\,.\,(2x + 1),$$

could not be integrated by this rule but the expression $(2x - 1).(2x + 1).(2x + 3)$ could.

33. Again since if $u_x = A_0 + A_1 x$,

$$\Delta \frac{a}{u_x u_{x+1} \ldots . u_{x+n-1}} = \frac{a}{u_{x+1} u_{x+2} \ldots . u_{x+n-1}} \left(\frac{1}{u_{x+n}} - \frac{1}{u_x}\right),$$

$$= \frac{-a}{u_x u_{x+1} \ldots . u_{x+n}} (u_{x+n} - u_x) = -\frac{naA_1}{u_x u_{x+1} \ldots . u_{x+n}} .$$

$$\therefore \Sigma \frac{a}{u_x . u_{x+1} \ldots . u_{x+n}} = -\frac{a}{u_x . u_{x+1} \ldots . u_{x+n-1}} . \frac{1}{nA_1} + C;$$

which furnishes a rule for integrating a fraction, whose denominator is the product of consecutive terms of an Arithmetical Progression, and numerator constant. It is "Efface the last factor, and divide by the number of factors remaining, and by the common difference, prefixing a negative sign."

34. This method will apply to the case where the numerator is a rational function, lower by two dimensions at least than the denominator. Thus, suppose the fraction to be

$$\frac{ax^{n-2} + bx^{n-3} + \ldots . + kx + l}{u_x \ldots . u_{x+n-1}},$$

assume,

$$ax^{n-2} + bx^{n-3} + \ldots + kx + l = a' + b'u_x + c'u_x u_{x+1} + \ldots + k'u_x u_{x+1} u_{x+2} \ldots u_{x+n-3}.$$

Then arranging the second member in terms proceeding by powers of x, and equating coefficients of like powers, we obtain $(n-1)$ equations for finding $(n-1)$ quantities, $a', b', c', \ldots . k'$, and the fraction is reduced to the form,

$$\frac{a'}{u_x \ldots . u_{x+n-1}} + \frac{b'}{u_{x+1} \ldots . u_{x+n-1}} + \ldots . + \frac{k'}{u_{x+n-2} u_{x+n-1}},$$

each term of which is integrable by the preceding rule.

If the dimension of the numerator were the same as that of the denominator, or differed from it only by a unit, we should arrive at a term $\frac{l'}{u_{x+n-1}}$, of which can be integrated only approximately.

35. To find the integral of a^x.

Since $$\Delta a^x = a^{x+1} - a^x = (a-1)a^x.$$

$$\therefore\ \Sigma a^x = \frac{a^x}{a-1} + C_{n-1}.$$

So $$\Sigma^2 a^x = \frac{\Sigma a^x}{a-1} + \Sigma C_{n-1} = \frac{a^x}{(a-1)^2} + C_{n-1}x + C_{n-2};$$

Similarly $$\Sigma^n a^x = \frac{a^x}{(a-1)^n} + (C_0 + C_1 x + \ldots . C_{n-1}x^{n-1}).$$

36. To find the integral of $\log v_x$.

Since

$$\Delta \log v_1 v_2 \ldots . v_{x-1} = \log v_1 v_2 \ldots . v_x - \log v_1 v_2 \ldots . v_{x-1} = \log v_x.$$

$$\therefore\ \Sigma \log v_x = \log v_1 v_2 \ldots . v_{x-1} + \log C;$$

or as we shall write it,

$$\Sigma \log v_x = \log CPv_{x-1}.$$

where Pv_x denotes the product of all the successive values of v_x, from some fixed term v_1, (or in general v_r) to v_x.

37. The integrals of trigonometrical functions are readily found by substituting their exponential values.

Thus $$2\Sigma \cos x\theta = \Sigma\left(e^{x\theta\sqrt{-1}} + e^{-x\theta\sqrt{-1}}\right) = \frac{e^{x\theta\sqrt{-1}}}{e^{\theta\sqrt{-1}} - 1} + \frac{e^{-x\theta\sqrt{-1}}}{e^{-\theta\sqrt{-1}} - 1},$$

$$= \frac{e^{(x-1)\theta\sqrt{-1}} - e^{x\theta\sqrt{-1}} + e^{-(x-1)\theta\sqrt{-1}} - e^{-x\theta\sqrt{-1}}}{2 - \left(e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}\right)} = \frac{2\cos(x-1)\,\theta - 2\cos x\theta}{2 - 2\cos\theta};$$

$$\therefore\ \Sigma \cos x\theta = \frac{\cos(x-1)\,\theta - \cos x\theta}{2(1-\cos\theta)}.$$

38. To find the n^{th} integrals of $\sin(ax+\beta)$, $\cos(ax+\beta)$.

Writing $-n$ for n in the results in Art. 10, we have, since we have shewn Δ^{-n} and Σ^n to be equivalent symbols,

$$\Sigma^n \sin(ax+\beta) = \frac{\sin\{(ax+\beta)-\frac{1}{2}n(\pi+a)\}}{(2\sin\frac{1}{2}a)^n};$$

$$\Sigma^n \cos(ax+\beta) = \frac{\cos\{(ax+\beta)-\frac{1}{2}n(\pi+a)\}}{(2\sin\frac{1}{2}a)^n}.$$

Integration by Parts.

39. Since $\qquad \Delta u_x v_x = u_x \Delta v_x + v_{x+1}\Delta u_x,$

$$\therefore\ \Sigma u_x \Delta v_x = u_x v_x - \Sigma(v_{x+1}\Delta u_x),$$

which is analogous to the form in the Integral Calculus,

$$\int dx\, u\frac{dv}{dx} = uv - \int dx\, v\frac{du}{dx}.$$

For v_x writing Σv_x, the expression becomes

$$\Sigma u_x v_x = u_x \Sigma v_x - \Sigma(\Delta u_x \Sigma v_{x+1}).$$

This formula admits of a very useful extension, as follows: since

$$\Delta^n u_x v_x = (DD'-1)^n u_x v_x,$$

where D refers to u_x, and D' to v_x, only: we have also,

$$\Delta^n u_x v_x = \{D(1+\Delta')-1\}^n u_x v_x = (\Delta + D\Delta')^n u_x v_x,$$

$$= \left\{\Delta^n + n\Delta^{n-1}.D.\Delta' + \frac{n.(n-1)}{1.2}\Delta^{n-2}.D^2.\Delta'^2 + \dots\right\} u_x v_x,$$

$$= v_x \Delta^n u_x + n.\Delta v_x \Delta^{n-1} u_{x+1} + \frac{n.(n-1)}{1.2}\Delta^2 v_x \Delta^{n-2} u_{x+2} + \dots$$

Since this is true for all values of n, suppose n negative. Then, since Δ^{-n} is equivalent to Σ^n,

$$\Sigma^n u_x v_x = v_x \Sigma^n u_x - n\Delta v_x \Sigma^{n+1} u_{x+1} + \frac{n.(n+1)}{1.2}\Delta^2 v_x \Sigma^{n+2} u_{x+2} - \&c.$$

Cor. Make $n = 1$,

$$\Sigma u_x v_x = v_x \Sigma u_x - \Delta v_x \Sigma^2 u_{x+1} + \Delta^2 v_x \Sigma^3 u_{x+2} - \&c.$$

This formula always enables us to integrate the product of two functions, one of which admits of successive integrations, and the other leads to zero, as the value of one of its successive differences.

40. To shew that $(D - a)^{-n} u_x = a^{x-n} \Sigma^n u_x a^{-x}$.

Since $$Da^x = a^{x+1} = a.a^x,$$

therefore in this case the symbol D is equivalent to the symbol a, prefixed to a^x.

Hence, $$\because \Delta^n u_x \alpha^x = (DD' - 1)^n u_x \alpha^x,$$

where D' refers to α^x, and D to u_x, *only*. Hence D' is equivalent to α, and we may write,

$$\Delta^n u_x \alpha^x = (D\alpha - 1)^n u_x \alpha^x = \alpha^n \left(D - \frac{1}{\alpha}\right)^n u_x \alpha^x.$$

Let $a = \frac{1}{\alpha}$;

then, $$\Delta^n u_x a^{-x} = a^{-n} (D - a)^n u_x a^{-x}.$$

Whence, $$(D - a)^n u_x = a^{x+n} \Delta^n u_x a^{-x}.$$

Since this is true for general values of n, we may suppose it negative, in which case Δ^{-n} is equivalent to Σ^n, and we have

$$(D - a)^{-n} u_x = a^{x-n} \Sigma^n . u_x a^{-x},$$

$$\text{if } n = 1, (D - a)^{-1} u_x = a^{x-1} \Sigma u_x a^{-x};$$

a theorem of great importance in the Solution of Linear Equations.

Calculus of Generating Functions.

41. If u_x be any function of x, and the series

$$\ldots\ldots + u_{-1} t^{-1} + u_0 + u_1 t + \ldots\ldots + u_{x-1} t^{x-1} + u_x t^x + u_{x+1} t^{x+1} + \ldots\ldots$$

be denoted by $\phi(t)$; then $\phi(t)$ is called the generating function of u_x, because by its expansion it generates the coefficients $u_0, u_1 \ldots . u_x$, annexed to the corresponding powers of t; it is expressed thus, $\phi(t) = Gu_x$.

Thus, since $\log(1-t)^{-1} = t + \frac{t^2}{2} + \frac{t^3}{3} + \ldots . + \frac{t^x}{x} + \ldots .$

$$\therefore \log(1-t)^{-1} = G\frac{1}{x}.$$

42. The following theorem is of great use in this branch of the subject.

If $\phi(t)$ be the generating function of u_x, or $\phi(t) = Gu_x$, and $\psi(t)$ any function of t, which can be expanded in integral powers of t, positive or negative, then

$$\phi(t)\,\psi(t) = G\psi\left(\frac{1}{D}\right)u_x,$$

$$\text{and } \phi(t)\,\psi\left(\frac{1}{t}\right) = G.\psi(D)u_x.$$

Let $\psi(t) = \ldots . + a_{-1}t^{-1} + a_0 + a_1t + a_2t^2 + \ldots . + a_xt^x + \ldots .$

Then, because

$$\phi(t) = \ldots . + u_{-1}t^{-1} + u_0 + u_1t + u_2t^2 + \ldots . + u_xt^x + \ldots .$$

we have for the coefficient of t^x, in the product $\psi(t).\phi(t)$,

$$\ldots . + a_{-1}u_{x+1} + a_0u_x + a_1u_{x-1} + a_2u_{x-2} + \ldots .$$

or $$(\ldots . + a_{-1}D + a_0 + a_1D^{-1} + a_2D^{-2} + \ldots .)u_x,$$

or $$\psi\left(\frac{1}{D}\right)u_x.$$

That is to say, $G\psi\left(\frac{1}{D}\right)u_x = \psi(t).\phi(t). = \psi(t)Gu_x$.

Similarly, $G\psi(D)u_x = \psi\left(\frac{1}{t}\right).\phi(t) = \psi\left(\frac{1}{t}\right)Gu_x$.

43. A variety of interesting results may hence be deduced.

Let $\psi(D) = D^{-n}$,

then $$G(D^{-n}u_x),\quad \text{or}\quad Gu_{x-n} = t^n.Gu_x.$$

So if $\psi(D) = D^n$,

then $$G(D^n u_x),\quad \text{or}\quad Gu_{x+n} = t^{-n}Gu_x.$$

Again,

Let $\psi(D) = (D-1)^n. = \Delta^n.$

then $$G\Delta^n u_x = \left(\frac{1}{t} - 1\right)^n Gu_x.$$

Writing $-n$ for n,

$$G\Sigma^n u_x = \left(\frac{1}{t} - 1\right)^{-n} Gu_x;$$

Again, let $$\psi(D) = (\log D)^n;$$

then $$G.(\log D)^n u_x = \left(\log\frac{1}{t}\right)^n Gu_x;$$

but $$\log D = \log e^{\frac{d}{dx}} = \frac{d}{dx},$$

$$\therefore\ G\frac{d^n}{dx^n}u_x = \left(\log\frac{1}{t}\right)^n Gu_x.$$

Hence $$G\int^n dx^n u_x = \left(\log\frac{1}{t}\right)^{-n} Gu_x.$$

Numbers of Bernouilli.

44. This is an appellation applied to distinguish certain coefficients which occur in the expansion of the function

$$\phi(t) = \frac{t}{e^t - 1},$$

and which are frequently employed in analysis.

The development can involve no odd powers of t above the first. For, since

$$\frac{-t}{e^{-t}-1} = \frac{te^t}{e^t-1} = t + \frac{t}{e^t-1};$$

$$\therefore\ \phi(-t) = t + \phi(t), \quad \text{and } \therefore\ \phi(-t) - \phi(t) = t,$$

so that when in the series for $\phi(t)$ we put $-t$ for t, and subtract, all the terms vanish, but that which involves t, which could not be if there were any higher odd powers than the first.

Again, the first term of the series is unity, and the second $-\frac{t}{2}$. For, by Maclaurin's theorem, the first term is $\phi(0)$, and since

$$\frac{t}{e^t-1} = \frac{t}{t+\frac{t^2}{1.2}+\ldots} = \frac{1}{1+\frac{t}{1.2}+\ldots}, \qquad \therefore\ \phi(0) = 1.$$

Again, if $$\phi(t) = 1 + At + \ldots.$$

$$\therefore\ \phi(-t) = 1 - At + \ldots.$$

and $$\phi(t) - \phi(-t) = 2At = -t, \quad \therefore\ A = -\tfrac{1}{2}.$$

Hence the series runs thus,

$$\frac{t}{e^t-1} = 1 - \tfrac{1}{2}t + \frac{B_1}{1.2}t^2 - \frac{B_3}{1.2.3.4}t^4 + \ldots + (-1)^{n+1}\frac{B_{2n-1}}{1.2\ldots 2n}t^{2n} + \ldots$$

and B_1, B_3, &c. are called Bernouilli's numbers.

45. The numerical values of the coefficients may be obtained as follows.

By Maclaurin's theorem, if $\frac{t}{e^t-1} = u$,

$$u = (u)_{t=0} + \left(\frac{du}{dt}\right)_{t=0} t + \left(\frac{d^2u}{dt^2}\right)_{t=0} \frac{t^2}{1.2} + \left(\frac{d^3u}{dt^3}\right)_{t=0} \frac{t^3}{1.2.3} + \ldots.$$

$\left(\frac{d^nu}{dt^n}\right)_{t=0}$ denoting the value of $\frac{d^nu}{dt^n}$, when in it t is put $= 0$.

Hence, therefore,

$$B_1 = \left(\frac{d^2u}{dt^2}\right)_{t=0}, \quad -B_3 = \left(\frac{d^4u}{dt^4}\right)_{t=0}, \quad B_5 = \left(\frac{d^6u}{dt^6}\right)_{t=0}, \text{ and so on,}$$

and all the odd differential coefficients but the first vanish, when in them t is put $= 0$.

Now $u(e^t - 1) = t$, and differentiating n times,

$$e^t\left(1 + \frac{d}{dt}\right)^n u - \frac{d^nu}{dt^n} = 0, \text{ when } n \text{ is} > 1$$

$$= 1 \text{ when } n = 1.$$

Putting $t = 0$, and giving n successive values, we have

$$n = 1, \quad u = 1,$$

$$n = 2, \quad 1 + 2\frac{du}{dt} = 0, \quad \therefore \quad \frac{du}{dt} = -\tfrac{1}{2},$$

$$n = 3, \quad 1 + 3\frac{du}{dt} + 3\frac{d^2u}{dt^2} = 0, \quad \therefore \quad \frac{d^2u}{dt^2} = \tfrac{1}{6};$$

$$\frac{d^3u}{dt^3} = 0, \quad \frac{d^5u}{dt^5} = 0, \quad \frac{d^7u}{dt^7} = 0, \text{ \&c.} = 0;$$

$$n = 5, \quad 1 - \tfrac{5}{2} + \tfrac{10}{6} + 5\frac{d^5u}{dt^5} = 0, \quad \therefore \quad \frac{d^4u}{dt^4} = -\tfrac{1}{30},$$

$$n = 7, \quad 1 - \tfrac{7}{2} + \tfrac{7}{2} - \frac{7.6.5.4}{1.2.3.4} \cdot \tfrac{1}{30} + 7\frac{d^6u}{dt^6} = 0;$$

$$\therefore \quad \frac{d^6u}{dt^6} = \left(\tfrac{7}{6} - 1\right)\tfrac{1}{7} = \tfrac{1}{42};$$

$$n = 9, \quad \frac{d^8u}{dt^8} = -\tfrac{1}{30}, \text{ and so on.}$$

Hence, $B_1 = \tfrac{1}{6}$, $B_3 = \tfrac{1}{30}$, $B_5 = \tfrac{1}{42}$, $B_7 = \tfrac{1}{30}$,

and any function, which by any means can be brought into the form $\frac{t}{e^t - 1}$, can be expanded in terms involving Bernouilli's numbers.

46. To express $\Sigma^n u_x$ in terms of the differential coefficients, and integrals of u_x.

$$\Sigma^n u_x = \Delta^{-n} u_x = (D - 1)^{-n} u_x = (e^{\frac{d}{dx}} - 1)^{-n} u_x;$$

and therefore by the exponential theorem,

$$\Sigma^n u_x = \left(\frac{d}{dx} + \frac{1}{1.2}\frac{d^2}{dx^2} + \frac{1}{1.2.3}\frac{d^3}{dx^3} + \dots\right)^{-n} u_x.$$

47. When $n = 1$, we can expand in terms of Bernouilli's numbers as follows:—

$$\Sigma u_x = (e^{\frac{d}{dx}} - 1)^{-1} u_x = \left(\frac{d}{dx}\right)^{-1} \left\{\frac{\frac{d}{dx}}{e^{\frac{d}{dx}} - 1}\right\} u_x,$$

$$= \left(\frac{d}{dx}\right)^{-1} \left(1 - \tfrac{1}{2}\frac{d}{dx} + \frac{B_1}{1.2}\frac{d^2}{dx^2} - \frac{B_3}{1.2.3.4}\frac{d^4}{dx^4} + \dots\right) u_x,$$

$$= \left(\int dx - \tfrac{1}{2} + \frac{B_1}{1.2}\frac{d}{dx} - \frac{B_3}{1.2.3.4}\frac{d^3}{dx^3} + \dots\right) u_x,$$

$$= \int dx\, u_x - \tfrac{1}{2} u_x + \frac{B_1}{1.2}\frac{du_x}{du} - \frac{B_3}{1.2.3.4}\frac{d^3 u_x}{dx^3} + \dots,$$

a series of considerable use in integration.

48. To prove that if B_{2n-1} be the n^{th} of Bernouilli's numbers,

$$B_{2n-1} = (-1)^{n+1} \frac{\log D}{\Delta} .0^{2n}.$$

Since, employing a system of logarithms whose base is e,

$$\frac{t}{e^t - 1} = \frac{\log e^t}{e^t - 1},$$

therefore by Herschel's theorem, the coefficient of t^{2n} in the expansion of this function is

$$\frac{1}{1.2.\ldots.2n} \cdot \frac{\log(1+\Delta)}{\Delta} 0^{2n}.$$

Hence, $$B_{2n-1} = (-1)^{n+1} \frac{\log D}{\Delta} 0^{2n}:$$

expanding, we have

$$B_{2n-1} = (-1)^n . \left\{\frac{\Delta 0^{2n}}{2} - \frac{\Delta^2 0^{2n}}{3} + \frac{\Delta^3 0^{2n}}{4} - \&c.\right\}$$

This proposition connects Bernouilli's numbers with the differences of zero.

49. The Differential Calculus is a particular case of the Direct Method in Finite Differences, and the Integral Calculus a particular case of the Inverse Method in Finite Differences; hence, from results obtained in the latter sciences with an indeterminate increment for the principal variable, we may pass to results in the former, by making the increment indefinitely small.

50. We shall add a few examples to illustrate this.

(1) Given $\Delta \tan x\theta = \tan^{-1} \frac{h\theta}{1+(x+h).x\theta^2}$, find $\frac{d \tan^{-1} x\theta}{dx}$.

The problem consists in finding the limit of

$$\frac{1}{h} \tan^{-1} \frac{h\theta}{1+(x+h)\, x\theta^2},$$

when h is diminished indefinitely.

Let $$\frac{1}{h} \tan^{-1} \frac{h\theta}{1+(x+h)x\theta^2} = u,$$

$$\therefore \tan hu = \frac{h\theta}{1+(x+h).x\theta^2},$$

$$\therefore \frac{\tan hu}{hu} . u = \frac{\theta}{1+(x+h)\, x\theta^2},$$

Let h be supposed indefinitely small,

then limit $\dfrac{\tan hu}{hu} = 1$

and limit $u = \dfrac{\theta}{1 + x^2\theta^2} = \dfrac{d \tan^{-1} x\theta}{dx}$.

(2) $\Sigma x = \dfrac{x^2 - hx}{2h} + C$, find $\int dx\, x$.

It is the limit of Σhx, where h is supposed indefinitely small.

Hence limit $\Sigma hx =$ limit $\dfrac{x^2 - hx}{2} + C = \dfrac{x^2}{2} + C$.

Again, $\qquad \Sigma x^2 = \dfrac{x^3}{3h} - \dfrac{x^2}{2} + \dfrac{hx}{6} + C,$

$$\therefore \ \Sigma x^2 h = \frac{x^3}{3} - \frac{x^2}{2} h + \frac{x}{6} h^2 + C.$$

Passing to the limit $\int x^2 dx = \dfrac{x^3}{3} + C$.

(3) To deduce Taylor's theorem from the formula

$$u_{x+nh} = u_x + n\Delta u_x + \frac{n.(n-1)}{1.2} \Delta^2 u_x + \ldots.$$

This we do by making h evanescent in every term. Let $nh = k$, a finite quantity, then when h becomes evanescent, n is indefinitely increased.

Also limit of $\dfrac{\Delta u_x}{h} = \dfrac{du_x}{dx}$,

$$\text{limit of } \frac{\Delta^2 u_x}{h^2} = \text{limit of } \frac{\Delta . \left(\dfrac{\Delta u_x}{h}\right)}{h} = \frac{d^2 u_x}{dx^2},$$

and so on.

Now we may write the formula as follows:

$$u_{x+nh} = u_x + nh\,\frac{\Delta u_x}{h} + \frac{nh.(nh-h)}{1.2}\,\frac{\Delta^2 u_x}{h^2} + \ldots.$$

Hence, taking the limit of both sides when h is indefinitely diminished, since nh is always equal to k, we have, from the preceding results,

$$u_{x+k} = u_x + k\,\frac{du_x}{dx} + \frac{k^2}{1.2}\,\frac{d^2 u_x}{dx^2} + \ldots.$$

CHAPTER III.

EQUATIONS OF DIFFERENCES.

51. An Equation of Differences is a relation between a variable, any function of that variable, and the successive values or differences of that function.

Such equations may be thus represented:

$$F(x,\ u_x,\ u_{x+1} \ldots u_{x+n}) = 0,$$

or

$$f(x,\ u_x,\ \Delta u_x,\ \Delta^2 u_x \ldots \Delta^n u_x) = 0,$$

since the forms are convertible; for, as we have seen, u_{x+r} is expressible in terms of u_x and its (r) differences, and *vice versâ*, $\Delta^r u_x$ is expressible in terms of u_x and its (r) successive values.

The order of an equation of differences is determined by that of the highest difference or successive value it involves.

52. The complete integral of an equation of differences of the n^{th} order will contain n arbitrary constants.

This appears from considering the manner in which such an equation is formed.

Let

$$f(x,\ u_x,\ a) = 0$$

be an equation between the variable x, the function u_x, and the arbitrary constant a. The relation is true when $x + h$ is put for x, whence

$$f(x + h,\ u_{x+h},\ a) = 0.$$

It is unnecessary to add a constant after performing the integration indicated in the equation

$$\log Q_x = \Sigma \log A_x.$$

For suppose

$$\log Q_x = \Sigma \log A_x + \log C' = \log C'PA_{x-1}.$$

Then $$Q_x = C'PA_{x-1},$$

and $$u_x = C'PA_{x-1}\left\{\Sigma \frac{B_x}{C'.PA_x} + C''\right\}$$

$$= PA_{x-1}\left\{\Sigma \frac{B_x}{PA_x} + C'C''\right\},$$

which is the same result as before, $C'C''$ being equivalent to the single constant C.

Ex. 1. $$u_{x+1} - e^{2x-1}u_x = e^{x^2}.$$

Let $$e^{2x-1} = \frac{Q_{x+1}}{Q_x}.$$

Then, dividing by Q_{x+1},

$$\frac{u_{x+1}}{Q_{x+1}} - \frac{u_x}{Q_x} = \frac{e^{x^2}}{Q_{x+1}},$$

$$\text{or} \quad \Delta\left(\frac{u_x}{Q_x}\right) = \frac{e^{x^2}}{Q_{x+1}}.$$

But $$2x - 1 = \Delta \log Q_x,$$

$$\therefore \ (x-1)^2 = \log Q_x,$$

the constant being unnecessary.

Hence $$Q_{x+1} = e^{x^2},$$

and $$\Delta\left(\frac{u_x}{e^{(x-1)^2}}\right) = 1,$$

$$\therefore \ \frac{u_x}{e^{(x-1)^2}} = x + C,$$

$$\therefore \ u_x = e^{(x-1)^2}\{x + C\}.$$

Ex. 2. $$u_{x+1} - au_x = (2x+1)\,a^x.$$

Let $$a = \frac{Q_{x+1}}{Q_x};$$

then, dividing by Q_{x+1},

$$\frac{u_{x+1}}{Q_{x+1}} - \frac{u_x}{Q_x} = \frac{(2x+1)\,a^x}{Q_{x+1}},$$

or $$\Delta\,\frac{u_x}{Q_x} = \frac{(2x+1)\,a^x}{Q_{x+1}}.$$

But $$\Delta \log Q_x = \log a,$$

$$\therefore\ \log Q_x = (x-1)\log a,$$

no constant being necessary; i.e.

$$\log Q_x = \log a^{x-1},$$

$$\therefore\ Q_{x+1} = a^x.$$

Hence $$\Delta\,\frac{u_x}{a^{x-1}} = \frac{(2x+1)\,a^x}{a^x} = 2x+1,$$

$$\therefore\ \frac{u_x}{a^{x-1}} = \Sigma(2x+1) = x^2 + c^2,$$

c^2 being an arbitrary constant,

$$\therefore\ u_x = a^{x-1}\left\{x^2 + c^2\right\}.$$

Indirect Integrals of Equations of Differences.

54. We have stated above that equations of differences of the first order, when inexact, are formed by the elimination of a constant (a) between two equations, as

$$u_x = f(x, a) \quad\text{and}\quad u_{x+h} = f(x+h, a)\ldots\ldots\ldots(1).$$

If, however, we suppose (a) not to be a constant, but a function of x, and denote this function by a_x, then these equations become

$$u_x = f(x, a_x), \quad u_{x+h} = f(x+h, a_{x+h});$$

and (since we have made no supposition as to the form of the function) if we suppose it such that

$$f(x+h, a_{x+h}) = f(x+h, a_x) \ldots\ldots\ldots\ldots (2),$$

then the two equations become

$$u_x = f(x, a_x), \quad u_{x+h} = f(x+h, a_x) \ldots\ldots\ldots (3),$$

wherein a_x appears exactly as a did in (1), and therefore, by eliminating (a_x) from equations (3), we shall arrive at the same equation of differences as we did by eliminating (a) from (1).

Now the condition (2) is satisfied by supposing

$$a_{x+h} = a_x, \quad \text{or} \quad \Delta a_x = 0,$$

whence $a_x = a$, a constant; and, substituting this value, we have

$$u_x = f(x, a):$$

but there are other values of a_x which satisfy the condition (2); for since $a_{x+h} = a_x$ is a value of a_{x+h}, which satisfies equations (2), $a_{x+h} - a_x$ is, by the theory of equations, a factor of that equation, and hence we may divide out by it, and the result will be (if powers of the constant (a) higher than the first are involved in (1),) an equation of differences of the first order with reference to a_x: by solving which we shall obtain one or more values of a_x in terms of x and arbitrary constants, and the substitution of these in

$$u_x = f(x, a_x)$$

will give other and indirect integrals of the proposed equation of differences.

Indirect integrals of equations of differences are obtained on principles very similar to those on which singular solutions of differential equations are obtained, but they differ in this, that whilst the former may contain as many arbitrary constants as the complete integral from which they are derived, the latter can contain no arbitrary constant.

Ex. $$(x+1)\,u_x = xu_{x+1} + \frac{1}{m}(u_{x+1} - u_x)^2.$$

This may be written thus:

$$u_x = x\Delta u_x + \frac{1}{m}(\Delta u_x)^2.$$

Taking the difference, we get

$$\Delta u_x = \Delta u_x + (x+1)\,\Delta^2 u_x + \frac{1}{m}\,\Delta\,(\Delta u_x)^2,$$

or $$0 = (x+1)\,\Delta^2 u_x + \frac{1}{m}\,\Delta\,(\Delta u_x)^2.$$

This equation is satisfied by

$$\Delta u_x = a,$$

or $$u_x = ax + a'.$$

But, from the proposed equation,

$$u_x = ax + \frac{1}{m}\,a^2,$$

$$\therefore\ a' = \frac{1}{m}\,a^2,$$

and therefore $$u_x = ax + \frac{1}{m}\,a^2.$$

the complete integral with one arbitrary constant.

For the indirect integrals we must put a_x for a.

Then $$u_x = a_x x + \frac{1}{m}\,a_x^2,$$

$$\therefore\ u_{x+1} = a_{x+1}(x+1) + \frac{1}{m}(a_{x+1})^2.$$

But by the conditions of such integrals,

$$u_{x+1} = a_x(x+1) + \frac{1}{m}(a_x)^2.$$

Subtracting the two preceding equations,

$$0 = (x+1)(a_{x+1} - a_x) + \frac{1}{m}\{(a_{x+1})^2 - (a_x)^2\};$$

since $a_{x+1} - a_x = 0$ is not the solution,

$$\therefore \quad 0 = (x+1) + \frac{1}{m}(a_{x+1} + a_x)$$

is the equation for determining the solutions.

Now $a_{x+1} + a_x = -m(x+1)$ gives

$$-a_x = (-1)^{x-1}\Sigma m(x+1)(-1)^x$$

$$= m\frac{2x+1}{4} - C(-1)^x,$$

$$\therefore \quad u_x = \left\{C(-1)^x - \frac{(2x+1)m}{4}\right\}x + \frac{1}{m}\left\{C(-1)^x - \frac{(2x+1)m}{4}\right\}^2,$$

the indirect integral.

Linear Equations of Differences of all Orders.

55. The general form of such equations is

$$u_{x+n} + p_1 u_{x+n+1} + \ldots\ldots + p_n u_x = X,$$

where $p_1, p_2 \ldots p_n$ and X denote functions of x. To this form we may convert the following:

$$\Delta^n u_x + q_1 \Delta^{n-1} u_x + \ldots\ldots + q_n u_x = X,$$

where $q_1, q_2 \ldots q_n$ denote functions of x.

56. We shall first consider the case where the coefficients $p_1, p_2 \ldots p_n$ are constants.

According to the notation before introduced, we may write the first equation as follows:

$$D^n u_x + p_1 D^{n-1} u_x + \ldots . + p_n u_x = X;$$

or separating the symbols of operation from those of quantity, we may put it in the form,

$$(D^n + p_1 D^{n-1} + \ldots . + p_n)\, u_x = X,$$

or, for shortness, $f(D)\, u_x = X.$

Hence, performing on both sides the inverse process of $f(D)$,

$$u_x = \{f(D)\}^{-1} X = \{D^n + p_1 D^{n-1} + \ldots . + p_n\}^{-1} X.$$

Now, let the equation

$$z^n + p_1 z^{n-1} + \ldots . + p_n = 0,$$

(which is called the auxiliary equation to the proposed), be supposed to be made up of a series of unequal factors, such as $(z - a)$, and of a series of equal factors such as $(z - b)^p$, where (a) and (b) may be real or imaginary, this comprises every case that can occur in the composition of the equation; then, by the theory for the resolution of rational fractions, $\dfrac{1}{f(z)}$ may be resolved into a series of fractions of which the type is

$$\frac{M}{z - a},$$

where $$\frac{1}{M} = \left\{\frac{df(z)}{dz}\right\}_{z=a},$$

together with a series of fractions of the form

$$\frac{N_1}{(z - b)^p} + \frac{N_2}{(z - b)^{p-1}} + \ldots . + \frac{N_p}{z - b},$$

where $$N_{r+1} = \left\{\frac{1}{1.2\ldots r} \cdot \frac{d^r}{dz^r} \cdot \frac{(z - b)^p}{f(z)}\right\}_{(z=b)}.$$

Introducing the symbol of operation, since the method of resolution into partial fractions is independent of any properties of the symbol, except that it be subject to the commutative, distributive, and index laws, to which laws the symbol D is also subject, we have

$$
\begin{aligned}
u_x &= \{f(D)\}^{-1}X \\
&= \{M(D-a)^{-1} + \&c. + [N_1(D-b)^{-p} + N_2(D-b)^{-(p-1)} + \&c.] + \&c.\}\ X \\
&= \{M(D-a)^{-1}X + \&c.\} + \{N_1(D-b)^{-p}X + N_2(D-b)^{-(p-1)}X + \&c.\} + \ldots \\
&= \{Ma^{x-1}\Sigma a^{-x}X + \&c.\} + \{N_1 b^{x-p}\Sigma^p b^{-x}X + N_2 b^{x-p+1}\Sigma^{p-1}b^{-x}X + \&c.\} + \&c.
\end{aligned}
$$

by the foregoing theorem.

Each of the integrations will introduce an arbitrary constant, and the whole function thus introduced, which is called the complementary function, will be (making $X = 0$, and replacing the products of constant quantities by single constants),

$$(Aa^x + \ldots.) + (B_0 + B_1 x + \ldots. + B_r x^r)\, b^x + \&c.$$

57. The above is the simplest and most symmetrical form into which the solution of the equation can be brought, when a and b are real. When they are, one or both, imaginary, the result admits of some modification.

By the theory of equations, if $\alpha + \beta\sqrt{-1}$ be an imaginary root of the auxiliary equation, there will also be a conjugate root $\alpha - \beta\sqrt{-1}$. For the sake of generality, suppose this root repeated p times in the equation; then the solution will consist of a series of pairs of terms, the coefficient of one term in each pair being the same function of $\alpha + \beta\sqrt{-1}$ that the other is of $\alpha - \beta\sqrt{-1}$. By the theory of impossible quantities, these coefficients may be represented by $C + D\sqrt{-1}$, and $C - D\sqrt{-1}$, so that the corresponding terms introduced will be of the form

$$
\begin{aligned}
&(C + D\sqrt{-1})(\alpha + \beta\sqrt{-1})^{x-r}\Sigma^r(\alpha + \beta\sqrt{-1})^{-x}X, \\
&+ (C - D\sqrt{-1})(\alpha - \beta\sqrt{-1})^{x-r}\Sigma^r(\alpha - \beta\sqrt{-1})^{-x}X,
\end{aligned}
$$

and there will be a series of them obtainable by giving r all positive integer values up to p.

Let $$\alpha = \rho\cos\theta, \quad \beta = \rho\sin\theta,$$

then $$(\alpha + \beta\sqrt{-1})^{x-r} = \rho^{x-r}(\cos\theta + \sqrt{-1}\sin\theta)^{x-r},$$

$$= \rho^{x-r}\left\{\cos(x-r)\theta + \sqrt{-1}\sin(x-r)\theta\right\},$$

consequently the two sums are reduced to

$$2\rho^{x-r}\left\{\begin{array}{l}[C\cos(x-r)\theta - D\sin(x-r)\theta]\,\Sigma^r(\rho^{-x}X\cos x\theta)\\ +\,[C\sin(x-r)\theta + D\cos(x-r)\theta]\Sigma^r(\rho^{-x}X\sin x\theta)\end{array}\right\},$$

and the complementary function is a series of terms of the form

$$2\rho^{x-r}\left\{\begin{array}{l}[C\cos(x-r)\theta - D\sin(x-r)\theta](C_0 + C_1x + \ldots\ldots C_rx^r)\\ +\,[C\sin(x-r)\theta + D\sin(x-r)\theta](D_0 + D_1x + \ldots + D_rx^r)\end{array}\right\}.$$

58. The two last Articles exhibit the general solution of all linear equations of differences with constant coefficients. The process is simplified when X in the second member has certain forms.

Thus, if $X = 0$, the integration is dispensed with, and the solution is reduced to the process of finding the complementary function.

Again, if $X = a^{bx}$, or the equation is $f(D)\,u_x = a^{bx}$, then $u_x = \{f(D)\}^{-1}a^{bx}$.

But since $Da^{bx} = a^b.a^{bx}$, the symbol D is equivalent to the symbol of quantity a^b, which we may therefore interchange with D,

$$\therefore\ u_x = \frac{a^{bx}}{f(a^b)} + \text{complementary function.}$$

Again, if X consist of terms of the form Ax^m, that is to say, if it be a rational integral function of x, we must expand $f(D)$ or $f(1 + \Delta)$ in a series ascending by powers of Δ, so that we shall have

$$u_x = + \ldots\ldots + (A_0 + A_1\Delta + \ldots\ldots + A_m\Delta^m + \ldots\ldots)\,Ax^m + \ldots\ldots$$

and the series will be terminable, since m being the highest index terms after $\Delta^m x^m$ vanish.

Ex. 1. $$a^2 u_{x+2} - 2au_{x+1} + u_x = a^2 x,$$

or $$(a^2 D^2 - 2aD + 1) u_x = a^2 x;$$

$$\therefore \left(D - \frac{1}{a}\right)^2 u_x = x;$$

$$\therefore u_x = \left(D - \frac{1}{a}\right)^{-2} x = \left(\Delta + 1 - \frac{1}{a}\right)^{-2} x + \text{comp. function},$$

$$= \left\{\left(1 - \frac{1}{a}\right)^{-2} - 2\left(1 - \frac{1}{a}\right)^{-3} \Delta + \ldots\ldots\right\} x + \text{comp. function},$$

$$= \frac{a^2 x}{(a-1)^2} - \frac{2a^3}{(a-1)^3} + \text{comp. function},$$

which function is $\left(\frac{1}{a}\right)^{x-2} \Sigma^2 0$, or $\left(\frac{1}{a}\right)^{x-2} (C_0 + C_1 x)$;

$$\therefore u_x = \frac{a^2 x}{(a-1)^2} - \frac{2a^3}{(a-1)^3} + \left(\frac{1}{a}\right)^{x-2} (C_0 + C_1 x).$$

Ex. 2. A seed grain is planted, and when one year old produces the next year tenfold; and when two years old and upwards, produces annually eighteenfold. Every grain is planted as soon as it is produced, and the produce of each is according to the above law. Find the number of grains at the end of the x^{th} year.

Let u_x = number at the end of the x^{th} year.

Then in the $(x + 1)^{\text{th}}$ year there will be produced, 10 each by those in their second year, and 18 each by all the rest.

That is, $$u_{x+1} = u_x + 10(u_{x-1} - u_{x-2}) + 18\, u_{x-2},$$

or $$u_{x+1} - u_x - 10\, u_{x-1} - 8\, u_{x-2} = 0.$$

Let $u_{x-2} = v_x$,

then, $$v_{x+3} - v_{x+2} - 10\, v_{x+1} - 8\, v_x = 0,$$

or $$(D^3 - D^2 - 10\, D - 8)\, v_x = 0.$$

The roots of the auxiliary equation are 4, − 1, and − 2;

$$\therefore\ v_x = C_1 4^x + C_2(-1)^x + C_3(-2)^x,$$

$$\therefore\ u_x = v_{x+2} = C_1 4^x + C_2(-1)^x + C_3(-2)^x,$$

changing the constants.

Making $x = 0, 1, 2$ in succession,

$$u_0 = 0 = C_1 + C_2 + C_3,$$

$$u_1 = 1 = 4C_1 - C_2 - 2C_3,$$

$$u_2 = 11 = 16C_1 + C_2 + 4C_3,$$

whence, $$C_1 = \tfrac{7}{15},\quad C_2 = -\tfrac{9}{5},\quad C_3 = \tfrac{4}{3},$$

and $$u_x = \frac{7.4^x - 27(-1)^x + 20(-2)^x}{15}.$$

59. Our investigations hitherto have proceeded on the supposition that the increment is unity: we shall now shew how to solve the same class of equations on the hypothesis of an indeterminate increment h.

As before, let the general equation be thus represented,

$$f(D)u_x = X,$$

then $$u_x = \{f(D)\}^{-1}X.$$

Let $\{f(D)\}^{-1}$ be resolved by the theory for the decomposition of rational fractions into a series of terms,

$$M(D-a)^{-1} + \ldots. + N(D-b)^{-p} + \ldots.,$$

then $$u_x = \ldots.\ M(D-a)^{-1}X + \ldots. + N(D-b)^{-p}X + \ldots.,$$

a, b, &c. being roots of the auxiliary equation

$$f(z) = 0.$$

Now the theorem from which we find the value of $N.(D-b)^{-p}X$ is investigated on the supposition of an increment unity, and will therefore have to be modified when h is taken for the increment.

We have, as before,

$$\Delta u_x v_x = u_{x+h} v_{x+h} - u_x v_x = (DD' - 1) u_x v_x,$$

where D refers to u_x, and D' to v_x only.

Let $v_x = a^x$; then Dv_x or $v_{x+h} = a^h . a^x$, so that D' is equivalent to a^h.

Also $$\Delta^p u_x v_x = (DD' - 1)^p u_x v_x$$

becomes, when $v_x = a^x$,

$$\Delta^p u_x a^x = (Da^h - 1)^p . u_x . a^x = a^x (Da^h - 1)^p u_x,$$

$$= a^{x+ph}\left(D - \frac{1}{a^h}\right)^p u_x.$$

Let $$\frac{1}{a^h} = \alpha, \quad \therefore \quad a = \left(\frac{1}{\alpha}\right)^{\frac{1}{h}} = \alpha^{-\frac{1}{h}};$$

then $$(D - \alpha)^p u_x = \alpha^{\frac{x+ph}{h}} \Delta^p u_x \alpha^{-\frac{x}{h}};$$

changing p into $-p$, this becomes,

$$(D - \alpha)^{-p} u_x = \alpha^{\frac{x-ph}{h}} \Sigma^p u_x \alpha^{-\frac{x}{h}}.$$

Hence, applying this theorem to the general equation, we have

$$u_x = + \ldots\ldots + Ma^{\frac{x-h}{h}} \Sigma a^{-\frac{x}{h}} X + \ldots\ldots + Nb^{\frac{x-ph}{h}} \Sigma^p b^{-\frac{x}{h}} X + \ldots\ldots,$$

the complementary function being of the form

$$+ \ldots\ldots + Ma^{\frac{x-h}{h}} + \ldots\ldots + Nb^{\frac{x-ph}{h}}\left(C_0 + C_1 \frac{x}{h} + \ldots\ldots\right),$$

or $$+ M'a^{\frac{x}{h}} + \dots + N'b^{\frac{x}{h}}\left(C_0 + C_1\frac{x}{h} + \dots\right),$$

changing the constants.

Ex. $$u_{x+2h} - 2m\,u_{x+h} + (m^2 + n^2)\,u_x = 0.$$

This may be written

$$(D^2 - 2mD + m^2 + n^2)\,u_x = 0\,;$$

$$\therefore\ u_x = C_1(m + n\sqrt{-1})^{\frac{x}{h}} + C_2(m - n\sqrt{-1})^{\frac{x}{h}},$$

C_1 and C_2 being arbitrary constants.

Let $$m = \rho\cos\theta,\quad n = \rho\sin\theta\,;$$

$$\therefore\ u_x = C_1\rho^{\frac{x}{h}}(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{x}{h}} + C_2\rho^{\frac{x}{h}}(\cos\theta - \sqrt{-1}\sin\theta)^{\frac{x}{h}}.$$

$$= \rho^{\frac{x}{h}}\left\{(C_1 + C_2)\cos\frac{x}{h}\theta + \sqrt{-1}\,(C_1 - C_2)\sin\frac{x}{h}\theta\right\},$$

$$= \rho^{\frac{x}{h}}\left(A\cos\frac{x\theta}{h} + B\sin\frac{x\theta}{h}\right),$$

changing the constants.

60. The two following examples are added with a view to illustrate the analogy that exists between the methods and results in the Differential Calculus, and those in the Calculus of Finite Differences.

Ex. From the fundamental theorem

$$(D - a)^{-p}u_x = a^{\frac{x}{h}-p}\Sigma^p.u_x.a^{-\frac{x}{h}},$$

to pass to the corresponding result in the Differential Calculus.

We have, since $D = 1 + \Delta$,

$$\{\Delta - (a - 1)\}^{-p}u_x = a^{\frac{x}{h}-p}\Sigma^p u_x a^{-\frac{x}{h}}.$$

Let $a - 1 = ah$,

$$\therefore\ (\Delta - ah)^{-p} = (1 + ah)^{\frac{x}{h}-p}\,\Sigma^p u_x (1 + ah)^{-\frac{x}{h}},$$

$$\therefore\ \left(\frac{\Delta}{h} - a\right)^{-p} = (1 + ah)^{\frac{x}{h}-p}\,\Sigma^p h^p u_x (1 + ah)^{-\frac{x}{h}}.$$

Passing to the limiting case, when h is indefinitely small,

since limit of $\frac{\Delta}{h} = \frac{d}{dx}$; and limit of $\Sigma^p . h^p = \int^p dx^p$;

and limit of $(1 + ah)^{\frac{x}{h}} = e^{ax}$; and limit of $(1 + ah)^{-\frac{x}{h}} = e^{-ax}$:

$$\therefore\ \left(\frac{d}{dx} - a\right)^{-p} u_x = e^{ax}\int^p dx^p u_x e^{-ax}.$$

61. A Differential Equation is a particular case of an equation of Finite Differences, viz. the case where the indeterminate increment is indefinitely small; and from the solution of the latter, we may pass to the solution of the corresponding differential equation. The equation whose solution has just been obtained will furnish an instance.

This equation is, when expressed in terms of Δu_x, $\Delta^2 u_x$,

$$\Delta^2 u_x - 2(m-1)\Delta u_x + \{(m-1)^2 + n^2\}\, u_x = 0;$$

$$\therefore\ \frac{\Delta^2 u_x}{h^2} - 2\,\frac{m-1}{h}\,\frac{\Delta u_x}{h} + \frac{(m-1)^2 + n^2}{h^2}\, u_x = 0.$$

Let $m - 1 = m'h$, and $n = n'h$,

then $$\frac{\Delta^2 u_x}{h^2} - 2m'\,\frac{\Delta u_x}{h} + (m'^2 + n'^2)\, u_x = 0.$$

Passing to the limit, when h is indefinitely diminished.

$$\frac{d^2 u_x}{dx^2} - 2m'\,\frac{du_x}{dx} + m'^2 + n'^2 = 0.$$

The solution is,

$$u_x = \rho^{\frac{x}{h}}\left(A \cos\frac{x\theta}{h} + B \sin\frac{x\theta}{h}\right):$$

but $$\rho^2 = m^2 + n^2 = (1 + m'h)^2 + (n'h)^2;$$

$$\therefore\ \rho^{\frac{x}{h}} = \{(1 + m'h)^2 + (n'h)^2\}^{\frac{x}{2h}},$$

therefore limit of $\rho^{\frac{x}{h}}$ = limit of $\{1 + 2m'h + (m'^2 + n'^2)h^2\}^{\frac{x}{2h}}$,

$$= \text{limit of } \{1 + 2m'h\}^{\frac{x}{2h}},$$

$$= e^{m'x};$$

and $$\cos\frac{x\theta}{h} = \cos\left(\frac{x}{h}\tan^{-1}\frac{n}{m}\right) = \cos\left(\frac{x}{h}\tan^{-1}\frac{n'h}{1 + m'h}\right),$$

therefore limit of $\cos\frac{x\theta}{h} = \cos n'x$; so limit of $\sin\frac{x\theta}{h} = \sin n'x$;

$$\therefore\ u_x = e^{m'x}(A \cos n'x + B \sin n'x).$$

62. No general method has hitherto appeared for solving linear equations of differences with variable coefficients. In certain cases they can be reduced to equations with constant coefficients, in others they can be solved by finding the integral of an equation derived from the proposed, by destroying or modifying certain of its terms, and completing the solution by variation of parameters, as in differential equations, and a large class of equations be solved by the aid of the following theorem.

63. If a particular integral of a linear equation of any order, with no term independent of u_x, be known, the equation may be reduced to one of the same kind of the order immediately inferior.

Let the equation be reduced to the form

$$\Delta^n u_x + p_1 \Delta^{n-1} u_x + \dots + p_n u_x = 0,$$

and let $u_x = v_x$ be a particular integral.

Assume $$u_x = v_x \Sigma w_x.$$

Then $$\Delta^n (v_x \Sigma w_x) = (DD' - 1)^n (v_x \Sigma w_x),$$

where D refers to v_x, and D' to Σw_x, only.

Hence $\Delta^n (v_x \Sigma w_x) = \{D(1 + \Delta') - 1\}^n v_x \Sigma w_x = (\Delta + D\Delta')^n v_x \Sigma w_x$

$$= \left\{\Delta^n + n\Delta^{n-1} D\Delta' + \frac{n.(n-1)}{1.2} \Delta^{n-2} D^2 \Delta'^2 + \dots + D^n \Delta'^n\right\} v_x \Sigma w_x$$

$$= \Delta^n v_x \Sigma w_x + n\Delta^{n-1} v_{x+1} w_x + \frac{n.(n-1)}{1.2} \Delta^{n-2} v_{x+2} \Delta w_x + \dots + v_{x+n} \Delta^{n-1} w_x.$$

For n writing $(n-1)$, $(n-2)$...&c., in succession, and substituting the corresponding values of $\Delta^n u_x$, $\Delta^{n-1} u_x$, &c., in the proposed, it becomes

$$\Delta^n v_x \Sigma w_x + n\Delta^{n-1} v_{x+1} w_x + \frac{n(n-1)}{1.2} \Delta^{n-2} v_{x+2} \Delta w_x + \text{\&c.}$$
$$+ n\Delta v_{x+n-1} \Delta^{n-2} w_x + v_{x+n} \Delta^{n-1} w_x$$
$$+ p_1 \{\Delta^{n-1} v_x \Sigma w_x + (n-1) \Delta^{n-2} v_{x+1} w_x + \text{\&c.}$$
$$+ (n-1) \Delta v_{x+n-2} \Delta^{n-3} w_x + v_{x+n-1} \Delta^{n-2} w_x\} + \text{\&c.}$$
$$+ p_{n-1} (\Delta v_x \Sigma w_x + v_{x+1} w_x),$$
$$+ p_n (v_x \Sigma w_x).$$

Or $$(\Delta^n v_x + p_1 \Delta^{n-1} v_x + \text{\&c.} + p_n v_x) \Sigma w_x$$
$$+ \{n\Delta^{n-1} v_{x+1} + p_1 (n-1) \Delta^{n-2} v_{x+1} + \text{\&c.} + p_{n-1} v_{x+1}\} w_x$$
$$+ \left\{\frac{n(n-1)}{1.2} \Delta^{n-2} v_{x+2} + \frac{(n-1)(n-2)}{1.2} \Delta^{n-3} v_{x+2} + \dots + p_{n-2} v_{x+2}\right\} \Delta w_x$$
$$+ \text{\&c.}$$
$$+ (n\Delta v_{x+n-1} + p_1 v_{x+n-1}) \Delta^{n-2} w_x$$
$$+ v_{x+n} \Delta^{n-1} w_x.$$

Or, since the first line is zero, we get an equation of the form

$$\Delta^{n-1}w_x + q_1\Delta^{n-2}w_x + \&c. + q_{n-1}w_x = 0,$$

q_1, q_2, &c. being functions of x; and this equation is of an order immediately inferior to that of the proposed, and of the same kind.

64. Cor. Since $u_x = v_x\Sigma w_x$, $\therefore\ w_x = \Delta\dfrac{u_x}{v_x}$:

and if we know another particular integral of the proposed, $u_x = v_x'$, then

$$w_x = \Delta\frac{v_x'}{v_x}$$

is a particular integral of the last equation, which may be reduced now one order lower, that is to say, the solution of the original equation may be reduced to that of one, two orders lower: and, generally, if we know r particular integrals of a linear equation of the nth order, its solution may be reduced to one of the $(n-r)$th order.

Variation of Parameters.

65. This method, which is extensively employed in the solution of differential equations, may be applied also to equations of differences, as in the following example.

To solve $\qquad u_{x+1} - A_x u_x = B_x.$

The solution of $\qquad u_{x+1} - A_x u_x = 0,$

or $\qquad \dfrac{u_{x+1}}{u_x} = A_x,$

is $\qquad u_x = CPA_{x-1},$

where C is a parameter.

Now suppose C, in place of being constant, is a function of x, denoted by C_x, and assume that the integral of

$$u_{x+1} - A_x u_x = B_x$$

is of the same *form* as that of the modified equation; that is, assume the integral to be

$$u_x = C_x PA_{x-1},$$

where the form of C_x is to be determined from the condition of its being the solution.

Then $$u_{x+1} = C_{x+1} PA_x,$$

and $$u_{x+1} - A_x u_x = PA_x (C_{x+1} - C_x),$$

$$= B_x;$$

$$\therefore \ \Delta C_x = \frac{B_x}{PA_x},$$

and $$\therefore \ C_x = \Sigma \left(\frac{B_x}{PA_x} + C\right), \ C \text{ a constant.}$$

Whence, $$u_x = PA_{x-1} \left\{\Sigma \frac{B_x}{PA_x} + C\right\}.$$

Simultaneous Equations.

66. In these, instead of one equation of differences from which to determine a single function of x, we have two or more equations, each involving two or more functions of x, and their differences or successive values.

Where the equations of this class are linear with constant coefficients, the principles developed in the preceding pages can be applied to their solution. The theory of the method is as follows: since the symbols of operation employed in this Calculus are subject to the same laws of combination as those of quantity, they may be treated in the same manner. Hence we have only to separate the symbol of operation from its subject, and then proceed to eliminate as many as possible of the functions from between the given equations, treating the symbols of operation exactly as if they were symbols of quantity. The difficulty of elimination thus becomes reduced to that between ordinary and algebraical equations.

The following are illustrations of the method.

Ex. 1.
$$u_{x+1} = 2v_x + 12u_x,$$
$$v_{x+2} = 14v_x + 6v_{x+1} + 7u_{x+1}.$$

These may be thus written:

$$(D - 12)\,u_x - 2v_x = 0,$$
$$(D^2 - 6D - 14)\,v_x - 7Du_x = 0.$$

Eliminating u_x, regarding the coefficients as if they were symbols of quantity,

$$\{(D^2 - 6D - 14)(D - 12) - 14D\}\,v_x = 0;$$

reducing and simplifying,

$$(D^3 - 18D^2 + 44D + 168)\,v_x = 0,$$

or
$$(D - 6)(D + 2)(D - 14)\,v_x = 0.$$

Whence
$$v_x = a6^x + b(-2)^x + c(14)^x,$$

and
$$u_x = \frac{D^2 - 6D - 14}{7D}\,v_x$$
$$= -2a6^x + \tfrac{2}{7}b(-2)^x + c(14)^{x+1}.$$

Ex. 2.
$$u_{x+2} + 2v_{x+1} - 8u_x = a^x,$$
$$v_{x+2} - u_{x+1} - 2v_x = \frac{1}{a^x}.$$

Separating the symbols,

$$(D^2 - 8)\, u_x + 2Dv_x = a^x,$$

$$Du_x - (D^2 - 2)\, v_x = -\frac{1}{a^x}.$$

Eliminating v_x,

$$\{(D^2 - 8)(D^2 - 2) + 2D^2\}\, u_x = (D^2 - 2)\, a^x - 2D\,\frac{1}{a^x},$$

that is,
$$(D^2 - 2^2)^2\, u_x = (D^3 - 2)\, a^x - 2D\,\frac{1}{a^x}.$$

Whence, since $Da^x = a.a^x$,

$$u_x = \frac{a^2 - 2}{(a^2 - 4)^2}\, a^x - \frac{\frac{2}{a}\cdot\frac{1}{a^x}}{\left(\frac{1}{a^2} - 4\right)^2} + \text{comp. function.}$$

Now

$$(D^2 - 2^2)^{-2} = \tfrac{1}{16}(D + 2)^{-2} + \tfrac{1}{16}(D - 2)^{-2} + \tfrac{1}{32}(D + 2)^{-1} - \tfrac{1}{32}(D - 2)^{-1},$$

$$\therefore (D^2 - 2^2)^{-2} 0 = \tfrac{1}{16}(C_0 + C_1 x)(-2)^x + \tfrac{1}{16}(C_2 + C_3 x)\, 2^x + \tfrac{1}{32} C_4 (-2)^x - \tfrac{1}{32} C_5 2^x,$$

is the complementary function.

Also
$$v_x = \frac{a^x - (D^2 - 8)\, u_x}{2D}.$$

The number of constants, though apparently six, it will be seen on inspection, can be combined so as to give only four.

Application of the Calculus of Generating Functions to the Solution of Equations in Finite Differences.

67. The Calculus of Generating Functions has been applied to the solution of linear equations of differences. The method of solution is founded on the following theorem, demonstrated

in a former article:

$$G\,\psi(D)\,u_x = \psi\left(\frac{1}{t}\right) Gu_x.$$

Now let $f(D)\,u_x = X$ be an equation of differences: then

$$Gf(D)\,u_x = G(X).$$

Let $F(t)$ be the generating function of X; then, applying the previous theorem to the first member of the equation,

$$f\left(\frac{1}{t}\right) Gu_x = F(t),$$

$$\therefore\ Gu_x = \frac{F(t)}{f\left(\frac{1}{t}\right)}.$$

Let C_x be the coefficient of t in the expansion of $\dfrac{F(t)}{f\left(\frac{1}{t}\right)}$. Then $u_x = C_x$ is the solution of the proposed.

Ex. Let the equation be

$$u_{x+3} - 3u_{x+2} + 3u_{x+1} - u_x = x,$$

or
$$(D^3 - 3D^2 + 3D - 1)\,u_x = x;$$

i.e.
$$(D - 1)^3\,u_x = x,$$

$$\therefore\ G\,(D - 1)^3\,u_x = Gx,$$

or
$$\left(\frac{1}{t} - 1\right)^3 Gu_x = t.(1 - t)^{-2};$$

$$\therefore\ Gu_x = t^4\,(1 - t)^{-5}$$

$$= t^4\left\{1 + 5t + \frac{5.6}{1.2}\,t^2 + \ldots + \frac{5.6\ldots(5 + x - 1)}{1.2\ldots x}\,t^x + \ldots\right\},$$

On Partial Differences.

70. If $u_{x,y}$ denote a function of two variables x and y, the expression

$$u_{x+h,y+k} - u_{x,y}$$

is termed the total difference of the function $u_{x,y}$: and the expressions

$$u_{x+h,y} - u_{x,y},$$

$$u_{x,y+k} - u_{x,y},$$

are termed the partial differences of $u_{x,y}$ with respect to x and y respectively.

In like manner, the expression $u_{x+h,y+k}$ is termed the first total successive value of $u_{x,y}$, and the expressions $u_{x+h,y}$, $u_{x,y+k}$ are termed the first partial successive values of $u_{x,z}$ with reference to x and y respectively.

We shall use the following notation:

$$Du_{x,y} = u_{x+h,y+k}, \quad \Delta u_{x,y} = u_{x+h,y+k} - u_{x,y},$$

$$D_x u_{x,y} = u_{x+h,y}, \quad \Delta_x u_{x,y} = u_{x+h,y} - u_{x,y}$$

$$D_y u_{x,y} = u_{x,y+k}, \quad \Delta_y u_{x,y} = u_{x,y+k} - u_{x,y}.$$

Also Σ_x, Σ_y for summations with respect to x only, and y only, respectively.

The values of h and k are generally unity, unless the contrary be expressed. Since

$$u_{x+h,y+k} = Du_{x,y} = e^{h\frac{d}{dx} + k\frac{d}{dy}} u_{x,y},$$

therefore D is equivalent to $e^{h\frac{d}{dx} + k\frac{d}{dy}}$.

Also D_x, D_y are equivalent to $e^{h\frac{d}{dx}}$ and $e^{k\frac{d}{dy}}$ respectively,

$$\therefore\ D = D_x . D_y.$$

71. The symbols here introduced are subject to the three fundamental laws of algebraic combination; and hence a variety of relations between partial differences and successive values might be deduced, as in the case of ordinary differences.

72. It is obvious also that, as

$$\Delta u_{x,y} = (D - 1)\, u_{x,y},$$

when D and Δ are total operations referring to both variables,

$$\Delta^n u_{x,y} = \left\{D^n - nD^{n-1} + \frac{n.(n-1)}{1.2} D^{n-2} \ldots\right\} u_{x,y}$$

$$= u_{x+nh,\, y+nk} - n u_{x+(n-1)h,\, y+(n-1)k} + \ldots,$$

and, similarly,

$$D^n u_{x,y} = u_{x,y} + n\Delta u_{x,y} + \frac{n.(n-1)}{1.2} \Delta^2 u_{x,y} + \ldots$$

Equations of Partial Differences.

73. An Equation of Partial Differences is a relation between a function, its partial successive values or differences, and the variables or constants involved in the function.

We shall consider only Linear Equations with constant coefficients, and such as may be reduced to these.

By the method of Separation of Symbols the integration of linear equations of partial differences with constant coefficients is reduced to the same processes as apply to ordinary equations of differences of the same class.

There is an important distinction, however, to be observed with respect to the parameters in the two cases. As in the solution of ordinary equations of differences we continually meet with complementary functions of the form

$$Ca^x,$$

so in solving Equations of Partial Differences we shall find expressions of the form

$$D_y^x C_y,$$

in which the arbitrary function C_y takes the place of the arbitrary constant C.

Now we have before shewn that

$$D_y^x C_y = C_{y+x}.$$

Hence, in the solution of Equations of Partial Differences, arbitrary functions of binomials play the same parts as arbitrary constants multiplied by exponentials do in ordinary equations of differences.

Ex. 1. $u_{x+1,y} - u_{x,y+1} = x + y$.

This may be written,

$$(D_x - D_y)\, u_{x,y} = x + y,$$

whence,
$$u_{x,y} = D_y^{x-1}\Sigma_x D_y^{-x}(x + y).$$

The integration implied in the Σ_x will bring out not a constant, but a function of y, for the summation in Σ_x extends only to x.

Hence
$$u_{x,y} = D_y^{x-1}\Sigma_x y = D_y^{x-1}(xy + C_y),$$

$$= x(y + x - 1) + C_{y+x-1},$$

$$= x(y + x - 1) + C_{y+x},$$

where we change C_{y+x-1} into C_{y+x} for the sake of elegance.

Ex. 2. $u_{x+2,y} - 2u_{x+1,y+1} + u_{x,y+2} = x$.

This may be written,

$$(D_x^2 - 2D_x D_y + D_y^2)\, u_{x,y} = x,$$

or,
$$(D_x - D_y)^2 u_{x,y} = x.$$

Whence $u_{x,y} = D_y^{x-2}\Sigma_x^2.D_y^{-x}x$,

$$= D_y^{x-2}\left\{\Sigma_x \frac{(x-1)x}{2} + C_y'\right\},$$

$$= D_y^{x-2}\left\{\frac{(x-2)(x-1)x}{6} + C_y'x + C_y\right\},$$

$$= \frac{(x-2)(x-1)x}{6} + C_{y+x} + C'_{y+x}x,$$

changing the functions.

Ex. 3. $u_{x+2,y} - a^2u_{x,y+2} = 0$.

This may be written,

$$(D_x^2 - a^2D_y^2)u_{x,y} = 0.$$

Whence,

$$u_{x,y} = \tfrac{1}{2}D_x^{-1}\{(D_x + aD_y)^{-1} + (D_x - aD_y)^{-1}\}(0),$$

$$= \tfrac{1}{2}D_x^{-1}\{(-aD_y)^{x-1}C_y + (aD_y)^{x-1}C'_y\},$$

$$= \tfrac{1}{2}D_x^{-1}\{(-a)^{x-1}C_{y+x-1} + a^{x-1}C'_{y+x-1}\},$$

$$= \tfrac{1}{2}\{(-a)^{x-2}C_{y+x-2} + a^{x-2}C'_{y+x-2}\},$$

$$= \tfrac{1}{2}\{(-a)^{x}C_{y+x} + a^{x}C'_{y+x}\},$$

changing the functions.

Ex. 4. $u_{x+1,y+1} - bu_{x+1,y} - au_{x,y+1} + ab\,u_{x,y} = c^{x+y}$.

This may be written,

$$(D_xD_y - aD_y - bD_x + ab)u_{x,y} = c^{x+y},$$

or

$$(D_x - a)(D_y - b)u_{x,y} = c^{x+y}.$$

Whence, $u_{x,y} = (D_x - a)^{-1}(D_y - b)^{-1}c^{x+y}$,

$$= \frac{c^{x+y}}{(c-a)(c-b)} + \text{comp. function};$$

(since $D_x c^{x+y} = c.c^{x+y} = D_y c^{x+y}$, and therefore D_x and D_y are each equivalent to c),

$$\therefore\ u_{x,y} = \frac{c^{x+y}}{(c-a)(c-b)} + C_x b^y + C'_y a^x.$$

Ex. 5. $u_{x+1,y+n} - u_{x,y} = a^{x-y}$.

This may be written,

$$(D_x D_y^n - 1) u_{x,y} = a^{x-y},$$

or

$$D_y^n (D_x - D_y^{-n}) u_{x,y} = a^{x-y};$$

$$\therefore\ D_y^n u_{x,y} = D_y^{-n(x-1)} \Sigma_x D_y^{nx} a^{x-y},$$

$$= D_y^{-n(x-1)} \Sigma_x a^{(1-n)x-y},$$

$$= D_y^{-n(x-1)} \left\{ \frac{a^{(1-n)x-y}}{a^{1-n}-1} + C_y \right\},$$

$$u_{x,y+n} = \frac{a^{x-y-n}}{a^{1-n}-1} + C_{y-nx+n},$$

$$u_{x,y} = \frac{a^{x-y}}{a^{1-n}-1} + C_{y-nx}.$$

This result might have been obtained at once by remembering that D_x is equivalent to a, and D_y to a^{-1}.

74. The foregoing principles will enable us to effect the solution of the more general equation,

$$u_{x+n,y} + p_1 u_{x+n-1,y+1} + p_2 u_{x+n-2,y+2} + \dots + p_n u_{x,y+n} = V,$$

where the coefficients are constants, and V denotes a function of x and y.

The equation may be written as under,

$$(D_x^n + p_1 D_x^{n-1} D_y + p_2 D_x^{n-2} D_y^2 + \dots + p_n D_y^n) u_{x,y} = V.$$

Whence,

$$u_{x,y} = (D_x^n + p_1 D_x^{n-1} D_y + \dots\dots + p_n D_y^n)^{-1} V,$$

$$= \{(D_x D_y^{-1})^n + p_1 (D_x D_y^{-1})^{n-1} + \dots\dots + p_n\}^{-1} D_y^{-n} V.$$

The auxiliary equation,

$$z^n + p_1 z^{n-1} + \dots\dots + p_n = 0,$$

will in general be composed of unequal factors, real or imaginary, one of which may be represented by $z - a$, and of equal factors, real or imaginary, one series of which may be represented by $(z - b)^p$.

Therefore, introducing the symbols of operation, we shall have

$$u_{x,y} = \{(D_x D_y^{-1} - a) \dots . (D_x D_y^{-1} - b)^p \dots .\}^{-1} D_y^{-n} V,$$

and by the theory of partial fractions this may be put in the form,

$$u_{x,y} = \{M (D_x D_y^{-1} - a)^{-1} + \dots\dots$$

$$+ N_1 (D_x D_y^{-1} - b)^{-p} + N_2 (D_x D_y^{-1} - b)^{-(p-1)} + \dots\dots\} D_y^{-n} V;$$

$$= M (D_x - a D_y)^{-1} D_y^{-(n-1)} V + \dots\dots$$

$$+ N_1 (D_x - b D_y)^{-p} D_y^{-(n-p)} V + N_2 (D_x - b D_y)^{-(p-1)} D_y^{-(n-p+1)} V + \dots$$

Hence,

$$u_{x,y} = M (a D_y)^{x-1} \Sigma_x (a D_y)^{-x} D_y^{-(n-1)} V + \dots\dots$$

$$+ N_1 (b D_y)^{x-p} \Sigma_x^p (b D_y)^{-x} D_y^{-(n-p)} V + \dots\dots,$$

The complementary functions are supposed to be included under the signs of summation; to see their forms we need only write zero after those signs.

Therefore complementary function,

$$= M(aD_y)^{x-1}\Sigma_x(0) + \ldots.$$

$$+ N_1(bD_y)^{x-p}\Sigma_x^{\,p}(0) + \ldots.$$

$$= M(aD_y)^{x-1}C_y + \ldots.,$$

$$+ N_1(bD_y)^{x-p}({}_0C_y + {}_1C_y x + \ldots. + {}_{p-1}C_y x^{p-1}),$$

$$+ \ldots.$$

$$= Ma^{x-1}C_{y+x-1} + \ldots.,$$

$$+ N_1 b^{x-p}({}_0C_{y+x-p} + {}_1C_{y+x-p}x + \ldots.),$$

$$+ \ldots.$$

or changing the form of the functions,

$$= Ma^x C_{y+x} + \ldots.,$$

$$+ N_1 b^{x-p}({}_0C_{y+x} + {}_1C_{y+x}x + \ldots. + {}_{p-1}C_{y+x}x^{p-1}),$$

$$+ \ldots.$$

If the quantities corresponding to a ... b ... be imaginary, the results will have to be modified, as we have shewn in the case of ordinary equations of differences.

Of the Failure of Formulæ in the Inverse Processes of the Calculus of Finite Differences.

75. There are several instances in the Calculus of Finite Differences, as in the Differential Calculus in which a certain relation of the subject to an inverse operation makes the general formulæ fail. The method of arriving at the true value of the integral in these cases of failure is as follows:

since the function in this particular case becomes infinite, we so assume the arbitrary constant in the complementary function, as to make the formula for that particular value which gives rise to the discrepancy take the form $\frac{0}{0}$, the true value of which must be found by the usual rules. The constant is thus made negative and infinite, so that the difference of two infinite quantities may be finite.

We proceed to consider the two following examples:

Ex. 1. The expression $\Sigma a^x = \dfrac{a^x}{a-1} + c'$ fails when $a = 1$; required the true value of the integral.

Assume $$c' = c - \frac{1}{a-1}.$$

then $$\Sigma a^x = \frac{a^x - 1}{a-1} + c.$$

Now limit of $\dfrac{a^x - 1}{a-1}$ when $a = 1$ is same as limit of $\dfrac{x.a^{x-1}}{1}$, or its value is x;

$$\therefore\ \Sigma a^x = x + c \text{ when } a = 1.$$

Ex. 2. The solution of the equation,

$$u_{x+1} - au_x = b^x,$$

is $$u_x = \frac{b^x}{b-a} + c'a^x,$$

which fails when $b = a$ required the true value of u_x.

Assume, $$c' = c - \frac{1}{b-a},$$

then $$u_x = \frac{b^x - a^x}{b - a} + ca^x,$$

$$= xa^{x-1} + ca^x,$$

when $$b = a.$$

Ex. 3. $u_{x+2} - 2au_{x+1} + a^2u_x = b^x.$

The solution is

$$u_x = \frac{b^x}{(b - a)^2} + (c_0' + c_1'x)a^x,$$

which fails for $b = a$ required the true value of u_x.

Let $c_0' = c_0 - \frac{1}{(b - a)^2}$;

$$\therefore\ u_x = \frac{b^x - a^x}{(b - a)^2} + (c_0 + c_1'x)\,a^x,$$

$$\text{limit of } \frac{b^x - a^x}{(b - a)^2} = \text{limit of } \frac{xb^{x-1}}{2(b - a)} = \text{limit of } \frac{x.(x - 1).b^{x-2}}{2},$$

$$= \frac{x.(x - 1)}{2}\,a^{x-2},$$

$$\therefore\ u_x = \frac{x.(x - 1)}{2}\,a^{x-2} + (c_0 + c_1'x)\,a^x.$$

Functional Equations.

76. There is a large class of functional equations of which particular, and in some cases general, solutions can be obtained by the aid of the Calculus of Finite Differences. The following examples are added with a view to illustrate the method of applying it for this purpose.

Ex. 1. $F(a+x).F(a-x) = a^2 - x^2$.

To find a particular solution.

This may be written,

$$\frac{F(a+x)}{a+x} \cdot \frac{F(a-x)}{a-x} = 1,$$

or

$$f(a+x)f(a-x) = 1,$$

if

$$\frac{F(a+x)}{a+x} = f(a+x).$$

Let

$$a + x = u_{z+1},$$

$$a - x = u_z,$$

$$\therefore\ u_{z+1} + u_z = 2a;$$

$$\therefore\ (D+1)u_z = 2a.$$

Hence

$$u_z = a - C_z(-1)^z = a - x,$$

where C_z is a function of z, such that

$$\Delta C_z = 0;$$

C_z may therefore be an arbitrary function of $\sin 2\pi z$ and $\cos 2\pi z$; to get, however, a particular solution, suppose C_z a constant, and replace it by the letter C,

$$\therefore\ x = C(-1)^z.$$

Again, let

$$f(a+x) = v_{z+1},$$

$$f(a-x) = v_z,$$

$$\therefore\ v_{z+1}v_z = 1;$$

$$\therefore\ \log v_{z+1} + \log v_z = 0,$$

$$\therefore\ (D+1)\log v_z = 0,$$

$$\therefore \log v_z = C'(-1)^z = \frac{C'}{C}x,$$

$$= mx, \text{ suppose}$$

$$\therefore f(a-x) = v_z = e^{mx},$$

$$\therefore f(x) = e^{m(a-x)},$$

$$\therefore F(x) = x.f(x) = x.e^{m(a-x)},$$

a result which may easily be verified.

Ex. 2. $f\left(\frac{1+x}{1-x}\right) - af(x) = 0.$

Let $x = u_z$ and $\frac{1+x}{1-x} = u_{z+1}$.

Then eliminating x, and multiplying up,

$$u_{z+1}.u_z - u_{z+1} + u_z + 1 = 0.$$

The solution of this is

$$x \text{ or } u_z = \tan\left(C + \tfrac{1}{4}\pi z\right),$$

C being an arbitrary constant.

Whence $z = \frac{4}{\pi}(\tan^{-1} x - C)$.

Again let $f(x) = v_z$,

and $$f\left(\frac{1+x}{1-x}\right) = v_{z+1};$$

$$\therefore v_{z+1} - a\,v_z = 0,$$

$$\therefore v_z = C'a^z,$$

or $$f(x) = C'a^{\frac{4}{\pi}(\tan^{-1}z - C)},$$

$$= C_1 a^{\frac{4}{\pi}\tan^{-1}z},$$

changing the constant.

Ex. 3. Required a particular solution of the equation,

$$\phi(x) + \phi\{x + \phi(x)\} = 0.$$

Let $$\phi\{x + \phi(x)\} = u_{z+1},$$

and $$\phi(x) = u_z;$$

$$\therefore\ u_{z+1} + u_z = 0,$$

the solution is,

$$u_z = c(-1)^z = \phi(x),$$

where c may be a function of z, which remains unaltered when $z + 1$ is put for z.

Again, let $$x + \phi(x) = v_{z+1},$$

$$x = v_z,$$

$$\therefore\ \phi(x) = c(-1)^z = v_{z+1} - v_z,$$

whence, $$v_z = -\frac{c(-1)^z}{2} + a = x,$$

c and a being supposed constants,

$$\therefore\ \phi(x) = c(-1)^z = 2(a - x),$$

a particular solution.

CHAPTER IV.

ON SERIES.

Integration of the General Term.

77. A series is defined to be a regular progression of terms, increasing or decreasing in magnitude according to a certain law.

The position of any term in a series is distinguished by the number of terms by which it is removed from some particular term selected for the purpose. This number is called the index of that term.

Hence when the law of formation of a series is given, and the index of any term assigned, the magnitude of that term can be computed, and thus the successive terms of the series produced in order.

78. Where we wish to regard a term with reference only to the manner in which it is produced, and not with reference to its numerical value, we put for the index a general symbol, as x, and the term thus exhibited is called the x^{th}, or general term of the series.

79. The law of formation of a series is represented by the function, which expresses the general term in terms of its index. Since the magnitude of every term depends only on its index, and the law of formation of the series, it follows that the form of this function does not change in passing from one term to another.

80. Wherever the general term of a series is an integrable function, we are enabled to find the sum of any number of terms taken in order, by means of the Calculus of Finite Differences.

There are two cases to consider: first, where the general term is given as an explicit function of the index; and secondly, where it is given in terms of the index by an equation of differences.

81. CASE 1. Let S_x be the sum of x terms of a series, whose general term is u_x.

Then $$S_x = u_1 + u_2 + \dots + u_x,$$

u_1 denoting the first term.

Also, $$S_{x+1} = u_1 + u_2 + \dots + u_x + u_{x+1}.$$

Hence $$S_{x+1} - S_x = u_{x+1} = \Delta S_x,$$

$$\therefore\ S_x = \Sigma u_{x+1} + C;$$

C denoting an arbitrary constant. Hence the sum of any number of terms ending with the general term u_x, is equal to the integral of the succeeding term together with a constant.

To determine the constant, make $x = 0$,

$$\therefore\ S_0 = 0 = (\Sigma u_{x+1})_{(x=0)} + C,$$

or $$0 = \Sigma u_1 + C, \text{ as we shall write it,}$$

$$\therefore\ S_x = \Sigma u_{x+1} - \Sigma u_1;$$

and in general the arbitrary constant will be determined by the term with which the series commences.

Ex. 1. To find the sum of x terms of the series

$$2 + 12 + 36 + 80 + \&c.$$

The general term is

$$u_x = x^3 + x^2 = x^3 - x + x^2 + x,$$

$$\therefore\ \Sigma u_x = \frac{(x-2)(x-1)x(x+1)}{4} + \frac{(x-1)x(x+1)}{3} + C.$$

Hence $$\Sigma u_{x+1} = \frac{(x-1)x(x+1)(x+2)}{4} + \frac{x(x+1)(x+2)}{3} + C,$$

$$\therefore\ \Sigma u_1 = C.$$

Hence $$S_x = \frac{(x-1)x(x+1)(x+2)}{4} + \frac{x(x+1)(x+2)}{3}.$$

Ex. 2. To find the sum of x terms of the series

$$1.3.5 + 3.5.7 + 5.7.9 + \ldots.$$

Here $$u_x = (2x-1)(2x+1)(2x+3),$$

$$\therefore\ \Sigma u_x = \frac{(2x-3)(2x-1)(2x+1)(2x+3)}{4.2};$$

$$\therefore\ \Sigma u_{x+1} = \frac{(2x-1)(2x+1)(2x+3)(2x+5)}{8},$$

$$\Sigma u_1 = -\tfrac{15}{8};$$

$$\therefore\ \text{sum to } x \text{ terms} = \frac{(2x-1)(2x+1)(2x+3)(2x+5)+15}{8}.$$

Ex. 3. To find the sum of x terms of the series

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \text{\&c.}$$

Here $$u_x = \frac{1}{(2x-1)(2x+1)};$$

$$\therefore\ \Sigma u_x = -\frac{1}{2(2x-1)},$$

$$\Sigma u_{x+1} = -\frac{1}{2(2x+1)} \quad\text{and}\quad \Sigma u_1 = -\tfrac{1}{2}.$$

Therefore sum to x terms $= \frac{1}{2} - \dfrac{1}{2(2x+1)}$,

and sum to $\infty = \frac{1}{2}$.

Ex. 4. To sum x terms of the series

$$\frac{1}{1.2.3.4} + \frac{3}{2.3.4.5} + \frac{5}{3.4.5.6} + \dots.$$

Here $u_x = \dfrac{2x-1}{x(x+1)(x+2)(x+3)}$

$$= \frac{2}{(x+1)(x+2)(x+3)} - \frac{1}{x(x+1)(x+2)(x+3)};$$

$$\therefore\ \Sigma u_x = -\frac{1}{2}\cdot\frac{2}{(x+1)(x+2)} + \frac{1}{3}\cdot\frac{1}{x(x+1)(x+2)} + C,$$

$$\therefore\ \Sigma_{x+1} = \frac{1}{3}\cdot\frac{1}{(x+1)(x+2)(x+3)} - \frac{1}{(x+2)(x+3)} + C,$$

and $\Sigma u_1 = \dfrac{1}{3}\cdot\dfrac{1}{1.2.3} - \dfrac{1}{2.3} + C.$

Therefore sum of x terms

$$= \frac{1}{3}\left\{\frac{1}{(x+1)(x+2)(x+3)} - \frac{1}{1.2.3}\right\} - \left\{\frac{1}{(x+2)(x+3)} - \frac{1}{2.3}\right\},$$

and sum to $\infty = \dfrac{2}{3}\cdot\dfrac{1}{2.3} = \dfrac{1}{9}$.

Ex. 5. To find the sum of x terms of the series

$$a \cos \theta + 2a^2 \cos 2\theta + 3a^3 \cos 3\theta + \ldots.$$

Here $$u_x = xa^x \cos x\theta,$$

and $\therefore \Sigma u_x = x\Sigma a^x \cos x\theta - \Sigma^2 a^{x+1} \cos (x+1)\theta$

$$= x\left(a^x . \frac{a \cos (x-1)\theta - \cos x\theta}{a^2 - 2a \cos \theta + 1}\right)$$

$$- \frac{a^4 . a^{x-1} \cos (x-1)\theta - 2a^2 . a^x \cos x\theta + a^{x+1} \cos (x+1)\theta}{(a^2 - 2a \cos \theta + 1)^2} + C;$$

$$\therefore \Sigma u_{x+1} = (x+1)\, a^{x+1} . \frac{a \cos x\theta - \cos (x+1)\theta}{a^2 - 2a \cos \theta + 1}$$

$$- a^{x+2} . \frac{a^2 \cos x\theta - 2a \cos (x+1)\theta + \cos (x+2)\theta}{(a^2 - 2a \cos \theta + 1)^2} + C,$$

$$\Sigma u_1 = a . \frac{a - \cos \theta}{a^2 - 2a \cos \theta + 1} - a^2 . \frac{a^2 - 2a \cos \theta + \cos 2\theta}{(a^2 - 2a \cos \theta + 1)^2} + C,$$

$$S_x = \frac{(x+1)\, a^{x+2} \{a \cos x\theta - \cos (x+1)\theta\} - a(a - \cos \theta)}{a^2 - 2a \cos \theta + 1}$$

$$- \frac{a^{x+2}\{a^2 \cos x\theta - 2a \cos (x+1)\theta + \cos (x+2)\theta\} - a^2(a^2 - 2a \cos \theta + \cos 2\theta)}{(a^2 - 2a \cos \theta + 1)^2}.$$

Recurring Series.

82. CASE II. In this case the first step towards the summation of the series is to solve the equation of differences which gives the form of the general term: to do this in all cases is not possible; where, however the equation is linear with constant coefficients, the solution can, as we have seen, be effected. Series corresponding to this last class of equations are called Recurring Series.

83. The equation of differences which determines the general term is in this case comprehended in the form,

$$u_{x+n} + p_1 u_{x+n-1} + \dots + p_{n-1} u_{x+1} + p_n u_x = 0,$$

and the series of coefficients which connects any term with those which precede it, is called the Scale of Relation.

Thus $$1 + p_1 + \dots + p_n,$$

is the scale of relation of the series, the general term of which is given by the equation written above.

84. If a series be stated to be recurring, the number n of terms in its scale of relation being given, the scale itself must be found by assuming an equation of connection,

$$u_{x+n} + p_1 u_{x+n-1} + \dots + p_n u_x = 0,$$

and by giving to u_x, $u_{x+1}, \dots$ the values corresponding to the given terms, so as to form a number of equations sufficient to determine the coefficients $p_1, p_2 \dots p_n$.

85. A recurring series may in most cases be considered as made up of a series of Geometrical Progressions.

For if $a_1, a_2 \dots a_n$, be the roots of the equation,

$$z^n + p_1 z^{n-1} + \dots + p_n = 0,$$

which is auxiliary to the equation of connection,

$$u_{x+n} + p_1 u_{x+n-1} + \dots + p_n u_x = 0,$$

and these roots be all real and different, the solution of the equation last written is

$$u_x = c_1 a_1^x + c_2 a_2^x + \dots + c_n a_n^x,$$

where $c_1, c_2 \dots c_n$, are arbitrary constants.

Hence, making $x = 1, 2, 3 \dots$ successively, and collecting the terms multiplied by the same constants, the series

$$u_1 + u_2 + \dots + u_x + \dots$$

is transformed into

$$\begin{aligned} & c_1(a_1 + a_1^2 + a_1^3 + \ldots . + a_1^x), \\ + & c_2(a_2 + a_2^2 + a_2^3 + \ldots . + a_2^x), \\ + & c_3(a_3 + a_3^2 + a_3^3 + \ldots . + a_3^x), \\ + & \ldots . \\ + & c_n(a_n + a_n^2 + a_n^3 + \ldots . + a_n^x); \end{aligned}$$

which is made up of a series of geometrical progressions.

86. Where however the auxiliary equation contains equal or impossible roots, the form of solution is altered, and the proposition no longer holds.

For suppose the equation to have r roots equal to a_1.

Then $u_x = (c_0 + c_1x + c_2x^2 + \ldots + c_{r-1}x^{r-1})\,a_1^x + c_{r+1}a^x_{r+1} + \ldots,$

and on making $x = 1, 2, 3 \ldots .$ successively, we do not resolve the series into progressions as above.

If there be imaginary roots, the solution takes the form,

$$u_x = (c_0 + c_1x + \ldots . + c_{r-1}x^{r-1})\,\rho^x \cos x\theta$$
$$+ (c_0' + c_1'x + \ldots . + c'_{r-1}x^{r-1})\,\rho^x \sin x\theta + \ldots . + c_na_n^x.$$

In this case therefore the series cannot be resolved into geometrical progressions.

87. The solution of the connecting equation will involve a number of arbitrary constants, to determine which, the general term must be made to coincide successively with a number of the given terms.

Ex. 1. To find the sum of x terms of the series,

$$2 + 16 + 68 + 244 + \ldots .$$

whose scale of relation is $1 + p_1 + p_2$.

Let the equation be,

$$u_{x+2} + p_1 u_{x+1} + p_2 u_x = 0$$

Hence, $$68 + 16p_1 + 2p_2 = 0,$$

$$244 + 68p_1 + 16p_2 = 0,$$

from the first,

$$544 + 128p_1 + 16p_2 = 0;$$

$$\therefore\ 300 + 60p_1 = 0,\quad \therefore\ p_1 = -5,$$

and $$2p_2 - 80 + 68 = 0,\quad \therefore\ p_2 = 6,$$

and $$u_{x+2} - 5u_{x+1} + 6u_x = 0;$$

or $$(D^2 - 5D + 6)u_x = 0,$$

or $$(D - 3)(D - 2)u_x = 0,$$

$$u_x = c_1 3^x + c_2 2^x.$$

Also $$u_1 = 2 = 3c_1 + 2\,c_2,$$

$$u_2 = 16 = 9c_1 + 4c_2,$$

whence $$c_1 = 4,\quad c_2 = -5.$$

Hence, $$u_x = 4.3^x - 5.2^x,$$

$$\therefore\ \Sigma u_x = 4.\frac{3^x}{2} - 5.\frac{2^x}{1},$$

$$\Sigma u_{x+1} = 2.3^{x+1} - 5.2^{x+1},$$

$$\Sigma u_1 = 2.3 - 5.2;$$

$$\therefore\ S_x = 6.(3^x - 1) - 10(2^x - 1),$$

the sum of x terms.

Ex. 2. To sum x terms of the series,

$$1 + 6 + 29 + 132 + \ldots.$$

whose scale of relation is $1 + p_1 + p_2$.

Let $$u_{x+2} + p_1 u_{x+1} + p_2 u_x = 0,$$
be the equation.

Then $$29 + 6p_1 + p_2 = 0,$$

$$132 + 29p_1 + 6p_2 = 0.$$

Multiplying the first equation by 6,

$$174 + 36p_1 + 6p_2 = 0,$$

whence, $$42 + 7p_1 = 0, \quad \therefore\ p_1 = -6,$$

and $$p_2 - 36 + 29 = 0, \quad \therefore\ p_2 = 7;$$

$$\therefore\ u_{x+2} - 6u_{x+1} + 7u_x = 0,$$

or $$(D^2 - 6D + 7)u_x = 0.$$

The solution of $$D^2 - 6D + 7 = 0,$$

or $$(D - 3)^2 = 9 - 7 = 2,$$

is $$D = 3 + \sqrt{2}, \text{ and } D = 3 - \sqrt{2}.$$

Hence, $$u_x = c_1(3 + \sqrt{2})^x + c_2(3 - \sqrt{2})^x.$$

Making $x = 1, 2$, successively,

$$u_1 = 1 = 3(c_1 + c_2) + \sqrt{2}(c_1 - c_2),$$

$$u_2 = 6 = 11(c_1 + c_2) + 6\sqrt{2}(c_1 - c_2),$$

from the first,

$$6 = 18(c_1 + c_2) + 6\sqrt{2}(c_1 - c_2),$$

$$\therefore\ 0 = c_1 + c_2,$$

$$\therefore \ \frac{1}{\sqrt{2}} = c_1 - c_2, \quad \therefore \ c_1 = -c_2 = \frac{1}{2\sqrt{2}},$$

$$\therefore \ 2\sqrt{2}\, u_x = (3+\sqrt{2})^x - (3-\sqrt{2})^x,$$

$$\therefore \ 2\sqrt{2}\, \Sigma u_{x+1} = \frac{(3+\sqrt{2})^{x+1}}{2+\sqrt{2}} - \frac{(3-\sqrt{2})^{x+1}}{2-\sqrt{2}},$$

$$2\sqrt{2}\, \Sigma u_1 = \frac{3+\sqrt{2}}{2+\sqrt{2}} - \frac{3-\sqrt{2}}{2-\sqrt{2}},$$

$$\therefore \ S_x = \Sigma u_{x+1} - \Sigma u_1 = \frac{1}{2\sqrt{2}}\left\{\frac{(3+\sqrt{2})^{x+1}-(3+\sqrt{2})}{2+\sqrt{2}} - \frac{(3-\sqrt{2})^{x+1}-(3-\sqrt{2})}{2-\sqrt{2}}\right\}.$$

Convergency and Divergency of Series.

88. A series is said to be convergent, when any number of its terms being taken, the sum of these terms has a finite quantity for its limit, when the number of terms so taken is indefinitely increased; and divergent in the contrary case.

89. In a convergent series, the sum of any number n of consecutive terms tends continually to zero as n is increased without limit.

Suppose $s_n = u_1 + u_2 + \dots + u_n$.

Then, $s_{n+m} = u_1 + u_2 + \dots + u_n + u_{n+1} + \dots + u_{n+m}$,

and $\therefore \ s_{n+m} - s_n = u_{n+1} + u_{n+2} + \dots + u_{n+m}$.

Now, as n is increased indefinitely, each of the quantities s_{n+m} and s_n approach to a finite limit, which limit is the same for both. Therefore $s_{n+m} - s_n$ approaches to zero as its limit; which proves the proposition.

Making $m = 1$, we see that u_{n+1} continually tends to zero as n is increased, which however is not a sufficient test of convergency.

Tests of Convergency.

90. (1) If u_n be the n^{th} term of the series,

$$u_1 + u_2 + \dots + u_n + \dots,$$

and the ratio $\frac{u_{n+1}}{u_n}$ approach to a finite quantity l as its limit, when n is indefinitely increased, the series will be convergent or divergent, according as l is less or greater than unity.

Let i be a number less than the difference between 1 and l, so that the two quantities $l - i$ and $l + i$ may be both less or both greater than 1, according as l is.

Then, by assigning to n a value large enough, the ratio $\frac{u_{n+1}}{u_n}$ may be made to lie between $l - i$ and $l + i$ for all superior values of n, and the terms of the series,

$$u_n,\ u_{n+1},\ u_{n+2},\ \&\text{c.},$$

will be confined within the corresponding terms of the series,

$$u_n,\ u_n(l - i),\ u_n(l - i)^2,\ \&\text{c.},$$

and
$$u_n,\ u_n(l + i),\ u_n(l + i)^2,\ \&\text{c.}$$

Now both the series last written, and therefore the intermediate series,

$$u_n,\ u_{n+1},\ u_{n+2},\ \&\text{c.},$$

will decrease indefinitely if u_n does so, as n is increased without limit, supposing l is less than 1, therefore the series is convergent in that case.

Also both series, and therefore the intermediate series will increase indefinitely if l is greater than 1, or the series is divergent.

If $l = 1$, the test gives no result.

91. (2) The series,

$$u_1 + u_2 + u_3 + \dots.$$

is convergent, or will become so, if the superior limit of $(u_n)^{\frac{1}{n}}$ be less than unity, when n is increased indefinitely, and divergent in the contrary case.

Let L denote the superior limit of $(u_n)^{\frac{1}{n}}$ when n is infinite. Suppose first L less than 1, and let k be a number between L and 1. Then, on increasing n without limit, $(u_n)^{\frac{1}{n}}$ will approach so near to L, as to become constantly less than k.

Hence we can give n so large a value, that for it and all superior values, $(u_n)^{\frac{1}{n}}$ is less than k, and therefore u_n less than k^n; consequently the series will conclude by always having its terms less than the corresponding terms of the progression,

$$k + k^2 + k^3 + \dots.,$$

which is convergent, since k is less than 1; therefore, *à fortiori*, the series proposed will end by becoming convergent.

In the next place, suppose L greater than 1. Let k be a quantity between 1 and L, so that L is greater than k, and k greater than 1. Then, when n is increased indefinitely, $(u_n)^{\frac{1}{n}}$ on approaching indefinitely near to L will become constantly greater than k. Hence we can give n so large a value, that for it and all superior values we shall have $(u_n)^{\frac{1}{n}}$ greater than k, and therefore u_n greater than k^n, and the proposed series will conclude by having its terms greater than the corresponding terms of the geometric progression,

$$k + k^2 + \dots. + k^n + k^{n+1} + \dots.,$$

which is divergent, k being greater than unity. Hence the proposed series will end by being divergent.

If $L = 1$, this test also gives no result.

92. The limits l and L determined in the two last Articles are identical, i.e.

$$\text{limit of} \left(\frac{u_{n+1}}{u_n}\right)_{n=\infty} = \text{limit of} \, (u_n)^{\frac{1}{n}}_{n=\infty}.$$

For, by taking m sufficiently large, each of the ratios

$$\frac{u_{m+1}}{u_m}, \; \frac{u_{m+2}}{u_{m+1}}, \; \ldots\ldots \; \frac{u_{m+n}}{u_{m+n-1}},$$

may be made to differ from l as little as we please: and therefore also the geometric mean between these ratios, viz.

$$\left(\frac{u_{m+n}}{u_m}\right)^{\frac{1}{n}},$$

may, by sufficiently increasing n, be made to differ from l by a quantity δ as small as we please; we shall have, therefore,

$$\{u_{n\left(1+\frac{m}{n}\right)}\}^{\frac{1}{n}} = (l \pm \delta) \, \{u_m\}^{\frac{1}{n}}.$$

Consequently, supposing n infinite, in which case δ vanishes,

$$\text{limit of} \, (u_n)^{\frac{1}{n}} = l;$$

$$\text{but limit of} \, (u_n)^{\frac{1}{n}} = L,$$

$$\therefore \; L = l.$$

Ex. 1. $e^x = 1 + \dfrac{x}{1} + \dfrac{x^2}{1.2} + \dfrac{x^3}{1.2.3} + \ldots\ldots + \dfrac{x^n}{1.2\ldots n} + \ldots\ldots$

$$\frac{u_{n+1}}{u_n} = \frac{x}{n+1} = 0 \text{ ultimately when } n \text{ is infinite;}$$

the series, therefore, is convergent for all values of x.

Ex. 2. $\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \ldots + (-1)^n \frac{x^{2n}}{1.2\ldots 2n} + \ldots.$

$$\frac{u_{n+1}}{u_n} = -\frac{x^2}{(2n+1)(2n+2)} = 0,$$

when n is sufficiently large. Therefore the series for $\cos x$ is always convergent. Similarly, the series for $\sin x$ is always convergent.

Ex. 3. The series for

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots . + (-1)^{n+1} \frac{x^n}{n} + \ldots.$$

is convergent or divergent, according as x is less or greater than unity. The same is true of the series for $\tan^{-1} x$, viz.

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots.$$

Ex. 4. $(1+x)^\mu = 1 + \mu x + \frac{\mu(\mu-1)}{1.2} x^2$

$$+ \frac{\mu(\mu-1)\ldots.(\mu-n+1)}{1.2\ldots.n} x^n + \ldots.,$$

$\therefore \frac{u_{n+1}}{u_n} = \frac{\mu - n}{n+1} x = -x$ ultimately when n is infinite;

and therefore if x is less than 1, the series is convergent.

93. (3) The series $u_1 - u_2 + u_3 - u_4 + \&c.$

is convergent, if the numerical value of its terms decreases without limit. For we may write it in either of the forms

$$u_1 - (u_2 - u_3) - (u_4 - u_5) - \&c.$$

or $$(u_1 - u_2) + (u_3 - u_4) + (u_4 - u_5) + \&c.,$$

which shew that the sum of the series is greater than $u_1 - u_2$, and less than u_1: hence it is convergent.

94. There is another test which it may be as well to notice; it is as follows: the series

$$u_1 + u_2 \dots\dots + u_n + \dots\dots$$

is convergent, or will become so, if the inferior limit of $\dfrac{\log u_n}{\log \frac{1}{n}}$ be greater than 1, when n is infinite; and divergent in the contrary case. The proof is very much the same as that of the test in Art. 91.

95. If the general term be imaginary, and therefore of the form

$$u_n = v_n + w_n \sqrt{-1},$$

we must assume

$$v_n = \rho_n \cos \theta_n \quad \text{and} \quad w_n = \rho_n \sin \theta_n,$$

$$\therefore \rho_n^2 = v_n^2 + w_n^2.$$

Then ρ_n is greater than v_n, and also greater than w_n. Also

$$u_n = \rho_n (\cos \theta_n + \sqrt{-1} \sin \theta_n),$$

and we may consider the series to be made up of two, viz.

$$(v_1 + v_2 + \dots\dots + v_n + \dots\dots) + \sqrt{-1}\,(w_1 + w_2 + \dots\dots + w_n + \dots\dots),$$

and the tests may be applied to them separately. Both will be convergent if the series of moduli,

$$\rho_1, \rho_2 \dots \rho_n,$$

form a convergent series.

On Interpolation.

96. The object of Interpolation is to insert amongst the terms of a given series new terms subject to the same law as those of the given series.

The practical use of this part of the subject is seen in cases where a series of values of a quantity being obtained, either from calculation or observation, it is necessary to insert between them other values, such as would have been obtained from similar observation or calculation without actually performing these.

The function which expresses the general term of the series is therefore what is required to be known: and it is to be found from a limited number of its numerical values. This cannot, however, be effected from these data, but if the terms given are sufficiently numerous, and differ but little from each other, the function can be found to a great degree of accuracy.

97. We proceed to consider the most usual cases of interpolation.

(1) Having given n values of a function $f(x)$,

$$u_1, u_2, u_3, \dots\dots u_n,$$

corresponding to n values of the variable

$$a, \quad a + h, \quad a + 2h, \dots\dots a + (n-1)h;$$

to find any intermediate value, as $f(a+k)$.

Let $$k = mh.$$
Then, $\because$

$$f(x + mh) = f(x) + m\Delta f(x) + \frac{m(m-1)}{1.2}\Delta^2 f(x) + \dots\dots$$

if we make $x = a$, and put $\frac{k}{h}$ for m, since the value of $f(x)$ in this case is u_1, we have

$$f(a+k) = u_1 + \frac{k}{h}\Delta u_1 + \frac{k(k-h)}{1.2.h^2}\Delta^2 u_1 + \dots\dots$$

But $\Delta u_1 = u_2 - u_1,$

$$\Delta^2 u_1 = u_3 - 2u_2 + u_1,$$

$$\&c. = \&c.,$$

$$\Delta^{n-1} u_1 = u_n - (n-1)\, u_{n-1} + \frac{(n-1)(n-2)}{1.2}\, u_{n-2} - \ldots \pm u_1.$$

Hence these quantities Δu_1, $\Delta^2 u_1$, ... are known, and on substituting them in the series for $f(a+k)$, we have the value required.

98. In using this formula, the easiest way of finding the values of Δu_1, $\Delta^2 u_1$, &c. is not from the above expansions, but as follows: write down the series of values u_1, u_2, u_3. . . .u_n, one under the other, and subtract each from the one under it, arranging the results one under the other in a separate column; the first of these is Δu_1. Perform the same process of subtraction with the terms in this column, and write the results as before, the first of these is $\Delta^2 u_1$, continuing the operation as long as there are any terms.

Ex. Having given the values of $\log_{10} 123$, $\log_{10} 124$, $\log_{10} 125$, $\log_{10} 126$; to find $\log_{10} 123.25$.

If $a = 123$, $k = .25 = \frac{1}{4}$, and $u_x = \log_{10} x$,

$u_a = 2.0899051$	Δ		
$u_{a+1} = 2.0934217$	35166	Δ^2	
$u_{a+2} = 2.0969100$	34883	-283	Δ^3
$u_{a+3} = 2.1003705$	34605	-278	5.

Now

$$u_{a+k} = u_a + k\Delta u_a + \frac{k(k-1)}{1.2}\, \Delta^2 u_a + \frac{k(k-1)(k-2)}{1.2.3}\, \Delta^3 u_a$$

terms after these being neglected.

Putting for k its value in the second member,

$$u_{a+k} = u_a + \tfrac{1}{4}\Delta u_a - \tfrac{3}{32}\Delta^2 u_a + \tfrac{7}{128}\Delta^3 u_a\,;$$

$$\begin{aligned}\therefore\ \log\left(123 + \tfrac{1}{4}\right) &= 2.0899051\\ &+\ .0008791\\ &+\ .0000027\\ &= 2.0907869.\end{aligned}$$

99. (2) Having given a series of consecutive and equidistant values of a function, where one is deficient to insert that one.

Let $u_1, u_2 \ldots, u_n$ be the values of the function, corresponding to the values

$$a, \quad a+h, \quad a+2h \ldots . a+(n-1)\,h$$

of the variable. In this case the terms given must be so numerous, and near each other, that $\Delta^{n-1}u_1$ is zero, or nearly so: this will almost always be the case in tabulated results.

$$\therefore\ \Delta^{n-1}u_1 = u_n - (n-1)\,u_{n-1} + \frac{(n-1)(n-2)}{1.2}\,u_{n-2} + \ldots \pm u_1 = 0:$$

from this equation any one of the values, as u_x, may be determined.

If two values are deficient, we must make another supposition, viz. that $\Delta^{n-2}u_1 = 0$, and we have thus two simple equations for the determination of the two quantities.

Ex. Given the logarithms of 121, 122, 124, 125; to find that of 123.

If $\quad a = 121$ and $u_x = \log_{10}x$,

$$\begin{aligned} u_a &= \log_{10}121 = 2.0827854,\\ u_{a+1} &= \log_{10}122 = 2.0863598,\\ u_{a+3} &= \log_{10}124 = 2.0934217,\\ u_{a+4} &= \log_{10}125 = 2.0969100\,;\end{aligned}$$

$\therefore$ putting $\Delta^4 u_a = u_{a+4} - 4u_{a+3} + 6u_{a+2} - 4u_{a+1} + u_a = 0$,

$$\begin{aligned}\therefore\ u_{a+2} &= \tfrac{1}{6}\{4(u_{a+3} + u_{a+1}) - (u_{a+4} + u_a)\}\\ &= \tfrac{1}{6}\{4(4.1797815) - (2.1796954)\}\\ &= \tfrac{1}{6}\{16.7191260 - 4.1796954\}\\ &= \tfrac{1}{6}\{12.5394306\}\\ &= 2.0899051.\end{aligned}$$

100. To interpolate any number of equidistant terms between every two consecutive terms of a given series.

Let the given series be $u_1, u_2 \ldots . u_n$, and let $(r - 1)$ equidistant terms be required to be inserted between every two of them.

The series will then become

$$u_1,\quad u_{1+\frac{1}{r}},\quad u_{1+\frac{2}{r}} \ldots . u_{1+\frac{r-1}{r}},\quad u_2,\quad u_{2+\frac{1}{r}},\quad u_{2+\frac{2}{r}} + \ldots .$$

Now $$u_{1+\frac{p}{r}} = u_1 + \frac{p}{r}\Delta u_1 + \frac{p.(p-r)}{1.2r^2}\Delta^2 u_1 + \ldots .$$

and therefore by making $p = 1, 2, \ldots .(r-1)$, in succession, since the values of Δu_1, $\Delta^2 u_1 \ldots$ are known from the given series, we get $(r - 1)$ terms between u_1 and u_2. Next, taking p from $r + 1$ to $2r - 1$, we get $(r - 1)$ terms between u_2 and u_3, and so on.

Lagrange's Formula of Interpolation.

101. By the aid of this formula the function can be found which for any number (n) of values $a_1, a_2 \ldots . a_n$ of the variable, proceeding according to no regular law, assumes the values

$$u_1,\ u_2,\ u_3, \ldots .\ u_n.$$

Let $$u_x = A_1 + A_2 + \ldots\ldots + A_n,$$

where $A_1, A_2, \ldots\ldots A_n$ are each functions of x; the form of u_x is in no wise restricted by this assumption.

Then, when $x = a_1$, the second member must reduce itself to u_1, in order to which all its terms must contain the factor $x - a_1$, except one, as A_1 for instance. Similarly all the terms, except one, as A_2 for instance, must contain the factor $x - a_2$, and so on. Hence each of the terms will contain all except one of the factors $x - a_1, x - a_2, \ldots\ldots x - a_n$, and that one different for each term.

Whence we gather that the form of u_x is,

$$u_x = A_1'(x - a_2)(x - a_3)\ldots(x - a_n) + A_2'(x - a_1)(x - a_3)\ldots(x - a_n),$$
$$+ \ldots\ldots$$

But when $x = a_1$, u_x becomes u_1, and hence

$$u_1 = A_1'(a_1 - a_2)(a_1 - a_3) \ldots\ldots (a_1 - a_n).$$

Similarly,

$$u_2 = A_2'(a_2 - a_1)(a_2 - a_3) \ldots\ldots (a_2 - a_n),$$

$$\&c. = \&c.,$$

$$u_n = A_n'(a_n - a_1)(a_n - a_2) \ldots\ldots (a_n - a_{n-1});$$

$$\therefore\ u_x = \frac{(x - a_2)(x - a_3) \ldots\ldots (x - a_n)}{(a_1 - a_2)(a_1 - a_3) \ldots\ldots (a_1 - a_n)}\, u_1,$$

$$+ \frac{(x - a_1)(x - a_3) \ldots\ldots (x - a_n)]}{(a_2 - a_1)(a_2 - a_3) \ldots\ldots (a_2 - a_n)}\, u_2,$$

$$+ \&c.$$

$$+ \frac{(x - a_1)(x - a_3) \ldots\ldots (x - a_{n-1})}{(a_n - a_1)(a_n - a_2) \ldots\ldots (a_n - a_{n-1})}\, u_n.$$

Ex. To find the function of x which, when $x = 4, 3, 2, 1$ shall assume the values, 5, 6, 7, 8,

$$u_x = 5\,\frac{(x-3)(x-2)(x-1)}{1.2.3} - 6\,\frac{(x-4)(x-2)(x-1)}{1.2}$$

$$+ 7\,\frac{(x-4)(x-3)(x-1)}{2.1.1} - 8\,\frac{(x-4)(x-3)(x-2)}{3.2.1}.$$

EXAMPLES.

I.

Direct Method of Differences.

1. $\Delta \dfrac{a^{2^x}+1}{a^{2^x}-1} = -\dfrac{2a^{2^x}}{a^{2^{x+1}}-1}.$

2. $\Delta 2^x \dfrac{a^{2^x}+1}{a^{2^x}-1} = 2^x \dfrac{a^{2^x}-1}{a^{2^x}+1}.$

3. $\Delta 2^x \sin \dfrac{\theta}{2^x} = 2^{x+2} \sin \dfrac{\theta}{2^{x+1}} \left(\sin \dfrac{\theta}{2^{x+2}}\right)^2.$

4. $\Delta \left(2^x \sin \dfrac{\theta}{2^x}\right)^2 = \left(2^{x+1} \sin \dfrac{\theta}{2^{x+1}}\right)^2.$

5. $\Delta \tan \dfrac{\theta}{2^x} = -\dfrac{\tan \dfrac{\theta}{2^{x+1}}}{\cos \dfrac{\theta}{2^x}}.$

6. $\Delta 2^x \tan \dfrac{\theta}{2^x} = -2^x \tan \dfrac{\theta}{2^x} \left(\tan \dfrac{\theta}{2^{x+1}}\right)^2.$

7. $\Delta (-2)^x \sin \dfrac{\theta}{2^x} = (-1)^{x+1} 2^{x+3} \sin \dfrac{\theta}{2^{x+2}} \left(\cos \dfrac{\theta}{2^{x+2}}\right)^3.$

8. $\Delta \dfrac{1}{2^x \sin \dfrac{\theta}{2^x}} = -2 \dfrac{\sin \dfrac{\theta}{2^{x+2}}}{2^x \sin \dfrac{\theta}{2^x}}.$

9. $\Delta \dfrac{1}{2^x \tan \dfrac{\theta}{2^x}} = \dfrac{\tan \dfrac{\theta}{2^{x+1}}}{2^{x+1}}.$

10. $$\Delta\left(\frac{1}{2^x\tan\dfrac{\theta}{2^x}}\right)^2=\frac{1}{2^{x+1}}-\left(\frac{\tan\dfrac{\theta}{2^{x+1}}}{2^{x+1}}\right)^2.$$

11. $$\Delta\frac{\cos\dfrac{\theta}{2^x}}{\left(2^x\sin\dfrac{\theta}{2^x}\right)^2}=\frac{\sin\dfrac{\theta}{2^{x+1}}}{\left(2^{x+1}\cos\dfrac{\theta}{2^{x+1}}\right)^3}.$$

12. $$\Delta\cot 2^x\theta=-\frac{1}{\sin 2^{x+1}\theta}.$$

13. $$\Delta\frac{\log(2\sin 2^x\theta)}{2^x}=-\frac{\log\tan 2^x\theta}{2^{x+1}}.$$

14. $$\Delta\frac{1}{\left(2^x\sin\dfrac{\theta}{2^x}\right)^2}=-\frac{1}{\left(2^{x+1}\cos\dfrac{\theta}{2^{x+1}}\right)^2}.$$

15. $$\Delta 2^x\tan^{-1}\frac{\theta}{2^x}=2^x\tan^{-1}\left(\frac{\theta^3}{4.2^{3x}+3.2^x\theta^2}\right).$$

16. From the expression

$$\Delta^n\sin x\theta=\left(2\sin\frac{h\theta}{2}\right)^n\sin\left\{x\theta+\frac{n}{2}(\pi+h\theta)\right\},$$

obtained on the hypothesis of an indeterminate increment h, deduce the value of $\dfrac{d^n\sin x\theta}{dx^n}$.

Differences of Zero.

1. Investigate the following relations:

$$\Delta^x0^x-\Delta^{x-1}0^x+\ldots\pm\Delta 0^x\mp 0^x=1;$$

and $$\frac{\Delta 0^x}{1}-\frac{\Delta^2 0^x}{2}\pm\ldots\pm\frac{\Delta^x 0^x}{x}=0$$

2. Shew that

$$\{\log(1+\Delta)\}^n f(\Delta)\,0^x = x(x-1)\ldots(x-n+1)f(\Delta)\,0^{x-n};$$

and that

$$f\{(1+\Delta)^n\}\,0^x = n^x f(1+\Delta)\,0^x.$$

3. Prove that

$$\left\{f(1+\Delta)+f\left(\frac{1}{1+\Delta}\right)\right\}0^{2x-1}=0;$$

and

$$\left\{f(1+\Delta)-f\left(\frac{1}{1+\Delta}\right)\right\}0^{2x}=0;$$

and thence deduce the expression

$$\frac{\Delta 0^{2x}}{2}-\frac{\Delta^2 0^{2x}}{2^2}+\frac{\Delta^3 0^{2x}}{2^3}+\ldots\pm\frac{\Delta^{2x}0^{2x}}{2^{2x}}=0.$$

4. Assuming $f(1+\Delta)$ in (3) to be

$$\int d\Delta\,\frac{\log(1+\Delta)}{\Delta};$$

prove that

$$\frac{\Delta 0^{2x}}{1^2}-\frac{\Delta^2 0^{2x}}{2^2}+\ldots\pm\frac{\Delta^{2x}0^{2x}}{(2x)^2}=0,$$

in all cases, except where $x=1$.

5. If $f_1(\Delta)$ and $f_2(\Delta)$ denote any two functions of Δ, then the following equation will hold good,

$$\{f_1(\Delta)f_2(\Delta)\}\,0^x = f_1(\Delta)f_2(\Delta')\,\{0+0'\}^x;$$

where in the second member the unaccented Δ is to be referred to the unaccented powers of 0, and the accented Δ to the accented powers.

Hence shew that

$$\{(1+\Delta)^n f(\Delta)\}\,0^x = f(\Delta)(n+0)^x.$$

6. Prove the following formula for changing the independent variable from x to e^x,

$$\left(y\,\frac{d}{dy}\right)^n=\frac{\Delta 0^n}{1}\,y\,\frac{d}{dy}+\frac{\Delta^2 0^n}{1.2}\,y^2\,\frac{d^2}{dy^2}+\frac{\Delta^3 0^n}{1.2.3}\,y^3\,\frac{d^3}{dy^3}+\ldots$$

7. Shew that if $u_x = a + hx$,

$$(u_x)^n = (-1)^{n+1}\left\{\frac{\Delta 0^n}{1}h^{n-1}u_x - \frac{\Delta^2 0^n}{1.2}h^{n-2}u_x u_{x+1} + \frac{\Delta^3 0^n}{1.2.3}h^{n-3}u_x u_{x+1} u_{x+2} - \dots\right\}$$

8. Investigate the expression,

$$\Delta^{-n}0^x = \frac{1.2\dots x}{1.2\dots(x+n)}\left\{\frac{\log(1+\Delta)}{\Delta}\right\}^n 0^{x+n},$$

and interpret its meaning.

II.

Inverse Method of Differences.

1. Find $\Sigma\left(\dfrac{1}{x^2 - a^2}\right)$, and $\Sigma(x^5 + x^3 + 1)$.

2. Find $\Sigma\left(\dfrac{x}{x^2 - 3x + 2}\right)$, and $\Sigma\left\{\dfrac{x}{2^x(x-1)(x+1)}\right\}$.

3. Find $\Sigma a^x \sin^2 x\theta$, and shew that a particular value of $\Sigma^n(a^{-x}\cos x\theta)$ is

$$\frac{a^{n-x}(D^{-1} - a)^n \cos x\theta}{(1 - 2a\cos\theta + a^2)^n}.$$

4. Investigate $\Sigma\dfrac{(x+1)^2}{x(x+2)(x+3)(x+4)}$ and $\Sigma\dfrac{3x-4}{(4x^2-1)(2x-7)}$.

5. Find

$\Sigma(2x+3)^3$, $\Sigma(x^2+x)(2x^2-6)^3$, and $\Sigma(x^4+1)^2$.

6. Reduce $u_{x+1}u_{x+3}$ to a series of terms, each of which is the product of consecutive terms of the arithmetical progression of which $u_x = a + hx$ is the general term.

7. Apply the same process to $(u_x)^2(u_{x+1})^3$.

Calculus of Generating Functions.

1. Shew that

$$G\cos x\theta = \frac{t\cos\theta - t^2}{1 - 2t\cos\theta + t^2},$$

and deduce $G\sin x\theta$.

2. Prove that

$$G\tan^{-1}x = \frac{1}{2\log\left(\frac{1}{t}\right)}\int dt\left(\frac{t^{\sqrt{-1}}}{t}\int dt\,\frac{t^{-\sqrt{-1}}}{1-t} + \frac{t^{-\sqrt{-1}}}{t}\int dt\,\frac{t^{\sqrt{-1}}}{1-t}\right).$$

3. Determine the generating functions of $\log x$, x^n, and x^{-n}.

4. Shew that

$$Ge^{-ax}\int dx\,e^{ax}u_x = (a - \log t)^{-1}Gu_x,$$

and

$$Ga^{x-1}\Sigma a^{-x}u_x = (1 - at)^{-1}tGu_x.$$

5. Find u_x from the following relations:

$$Gu_x = \frac{1 - \sqrt{(1 - 4t)}}{2t},$$

$$Gu_x = f(e^t).$$

6. Having given that $Gu_x = F(t)$, shew that

$$G(1^n + 2^n + \ldots + x^n)u_x = \frac{1+\Delta}{\Delta}\{F(t + t\Delta) - F(t)\}\,0^n.$$

Numbers of Bernouilli.

1. Prove that B_{2x-1} being the x^{th} of these numbers,

$$\frac{1}{2+\Delta}\,0^{2x-1} = (-1)^x\,\frac{2^{2x}-1}{2x}\,B_{2x-1}.$$

2. Also shew that

$$B_{2x-1} = (-1)^{x+1}\left\{\frac{\Delta 0^{2x+1}}{1^2} - \frac{\Delta^2 0^{2x+1}}{2^2} + \ldots + \frac{\Delta^{2x+1}0^{2x+1}}{(2x+1)^2}\right\}.$$

III.

Equations of Differences.

1. Solve the following equations:

$$u_{x+1} - (x+1)u_x = 1.2 \ldots (x+1);$$

and

$$(x+1)^2(u_{x+1} - au_x) = a^x.$$

2. Integrate $u_{x+1} - cu_x = \sin x\theta,$

and

$$u_{x+1} - cxu_x = \cos x\theta.$$

3. Solve

$$u_{x+2} - a(x+2)u_{x+1} + b(x+1)(x+2)u_x = 0;$$

and

$$u_{x+2} + ap^{2x}u_{x+1} + bp^{4x}u_x = cq^x p^{x^2}.$$

4. Find the integral of

$$u_{x+2} + a\phi(x+1)u_{x+1} + b\phi(x)\phi(x+1)u_x = c,$$

and that of

$$\Delta u_x + \Delta^2 u_x = 42u_x.$$

5. The solution of

$$(x+3)^2 u_{x+2} - 2\frac{(x+2)^2}{x+1}u_{x+1} + \frac{(x+1)^2(x+2)}{x}u_x = 0,$$

is

$$u_x = C_1\frac{x}{x+1} + C_2\left(\frac{x}{x+1}\right)^2.$$

6. If

$$(u_{x+1})^2 = 4(u_x)^2\{(u_x)^2 + 1\},$$

$$u_x = \tfrac{1}{2}(C^{2^x} - C^{-2x}).$$

7. Solve $(u_x)^2 - 4u_x u_{x+1} + a(2u_x + u_{x+1}) = 0;$

and

$$(u_x)^2 + 2u_x u_{x+1} - a(4u_x - u_{x+1}) = 0.$$

8. Prove that

$$y = ax + a^2,$$

and

$$y = \{\tfrac{1}{4} - b(-1)^x\}^2 - \frac{x^2}{4},$$

are distinct solutions of the same equation in finite differences.

9. Find the indirect integral of

$$u_x = x\Delta u_x - \frac{1}{\Delta u_x}.$$

10. Solve $u_{x+3} - 6u_{x+2} + 11u_{x+1} - 6u_x = a^x + x^2$;

and $$u_{x+1} - x^2 u_x = (1.2\ldots x)^2.$$

11. Shew that the general integral of the equation

$$u_{x+2}u_{x+1}u_x = a(u_{x+2} + u_{x+1} + u_x),$$

is comprised in that of the equation

$$u_{x+3} = u_x,$$

and is $$u_x = A + B\alpha^x + C\alpha^{2x};$$

α being one of the imaginary cube roots of unity, and A, B, C being connected by the equation

$$A^3 + B^3 + C^3 = 3A(a + BC).$$

Also shew that

$$u_x = \sqrt{a}\tan\left(C_1 \cos\frac{2\pi x}{3} + C_2 \sin\frac{2\pi x}{3}\right)$$

satisfies the equation.

12. If $$u_{x+1}u_x - a_x(u_{x+1} - u_x) + 1 = 0,$$

$$u_x = \tan\left(C + \Sigma \tan^{-1}\frac{1}{u_x}\right).$$

Simultaneous Equations.

1. Find u_x and v_x from the following equations:

$$u_{x+1} - 2v_x = a^x,$$

$$v_{x+1} + 2u_x = a^{-x}.$$

2. Solve $$au_{x+1} - b(u_x + v_x) = 0,$$

$$a(u_{x+1} + v_{x+1}) + b(u_x - v_x) = 0.$$

3. Solve
$$u_{x+2} - 2v_{x+1} + u_x = 0,$$
$$v_{x+2} - 2u_{x+1} + v_x = 0.$$

4. Solve
$$u_{x+3} - 8v_x = 27u_{x+1},$$
$$v_{x+3} - u_x = v_{x+1}.$$

5. Solve
$$u_{x+2} - (u_{x+1} + v_{x+1}) = u_x,$$
$$v_{x+2} + (u_{x+1} - v_{x+1}) = v_x.$$

Equations of Partial and Mixed Differences.

1. Find $u_{x,y}$ in each of the following equations:
$$u_{x+2,y} - u_{x,y+1} = a^{x+y},$$
$$u_{x+1,y} - 2u_{x,y+2} = x + y.$$

2. The solution of the equation,
$$u_{x+1,y} - A_{x,y}u_{x,y+1} = B_{x,y},$$
is
$$u_{x,y} = P.A_{x-1,y}\left(D_y^{x-1}\Sigma_x \frac{B_{x,y-x}}{P.A_{x,y-x}} + C_{y+x}\right),$$
where $P.A_{x,y}$ denotes the product of all the successive values of $A_{x,y}$ obtained by giving x all successive values up to x, and C_{y+x} is an arbitrary function of $y + x$.

3. Integrate the equation
$$u_{x+1,y} - \frac{d}{dy}u_{x,y} = 0.$$

4. The solution of
$$\Delta_x u_{x,y} = a\frac{d}{dy}u_{x,y},$$
is
$$u_{x,y} = a^x e^{-\frac{y}{a}}\left(\frac{d}{dy}\right)^x \phi(y),$$
ϕ denoting an arbitrary function.

5. If
$$u_{x+2,y} - a\frac{d}{dy}u_{x+1,y} + b\frac{d^2}{dy^2}u_{x,y} = 0;$$
$$u_{x,y} = \alpha^x\left(\frac{d}{dy}\right)^x\phi_1(y) + \beta^x\left(\frac{d}{dy}\right)^x\phi_2(y),$$
where α and β are the roots of $z^2 - az + b = 0$.

6. The integral of
$$\Delta_x^2 u_{x,y} - a\Delta_x\frac{d}{dy}u_{x,y} + b\frac{d^2}{dy^2}u_{x,y} = 0$$
is
$$u_{x,y} = \alpha^x e^{-\frac{\alpha}{y}}\left(\frac{d}{dy}\right)^x\phi_1(y) + \beta^x e^{-\frac{\beta}{y}}\left(\frac{d}{dy}\right)^x\phi_2(y),$$
where α and β are the same as in the preceding.

7. If $u_{x,y,z} - a\left(\frac{d}{dy}\right)^m u_{x-1,y,z} - b\left(\frac{d}{dy}\right)^n u_{x-2,y,z} = 0,$
$$u_{x,y,z} = a^x\left(\frac{d}{dy}\right)^{mx}\phi(y) + b^{\frac{x}{2}}\left(\frac{d}{dz}\right)^{\frac{nx}{2}}\psi(z).$$

Functional Equations.

1. Find a particular solution of the following:
$$f\left(\frac{1}{x}\right) + f(x) = a.$$

2. If
$$f(x) + af(-x) = x^n,$$
$$f(x) = \frac{1-(-1)^n a}{1-a^2}x^n.$$

3. If
$$f(x) - af\left(\frac{1}{x}\right) = e^x,$$
$$f(x) = \frac{e^x + ae^{\frac{1}{x}}}{1-a^2}.$$

4. If
$$\{f(x)\}^2 . f\left(\frac{1-x}{1+x}\right) = c^2 x,$$
$$f(x) = \left(\frac{1+x}{1-x} c^2 x^2\right)^{\frac{1}{3}}.$$

5. If
$$f(1+x) + f(1-x) = 1 - x^2,$$
$$f(x) = \frac{2x - x^2}{2} + \phi(x) + \phi(2-x).$$

IV.

On Series.

1. If $1.2.4 + 2.3.5 + 3.4.6$, to x terms, $= S_x$,

then
$$S_x = \frac{x(x+1)(x+2)(3x+13)}{12}.$$

2. If $1^4 + 3^4 + 5^4 \ldots + (2x-1)^4 = S_x$,

then
$$S_x = \frac{x(4x^2-1)(12x^2-7)}{15}.$$

3. If $\frac{1}{\sqrt{2}(1+\sqrt{2})} + \frac{1}{(1+\sqrt{2})(2+\sqrt{2})} + \frac{1}{(2+\sqrt{2})(3+\sqrt{2})}$
$+ \ldots$ to x terms, $= S_x$,

then
$$S_x = \frac{x}{\sqrt{2}(x+\sqrt{2})}.$$

4. If $1.3^2 + 3.5^2 + 5.7^2 + \ldots$ to x terms, $= S_x$,

then
$$S_x = \frac{x(6x^3 + 16x^2 + 9x - 4)}{3}.$$

5. If $\frac{3}{5.7} - \frac{9}{7.29} + \frac{27}{29.79} - \frac{81}{79.245} + \ldots$ to x terms, $= S_x$,

then
$$S_x = \frac{1}{20} - \frac{1}{4\{2 + (-3)^{x-1}\}}.$$

6. The series whose $(x+1)^{\text{th}}$ term is

$$\frac{(p+qx)\,t^x}{u_x \,.\, u_{x+1} \ldots u_{x+m-1}}, \text{ where } u_x = a + hx,$$

has for its sum

$$S_x = \frac{q}{(t-1)\,h}\left\{\frac{t^x}{u_x.u_{x+1}\ldots u_{x+m-2}} - \frac{1}{u_0u_1\ldots u_{m-2}}\right\};$$

when the following equation of condition holds good,

$$\frac{p}{q} = \frac{a}{h} - \frac{m-1}{t-1}.$$

7. Thus if $\frac{1^2.4}{2.3} + \frac{2^2.4^2}{3.4} + \frac{3^2.4^3}{4.5}$ to x terms, $= S_x$,

then $$S_x = \frac{2}{3} + \frac{4}{3}\,\frac{x-1}{x+2}\,.\,4^x.$$

8. If $1\cos^2\theta + 2\cos^2 2\theta + 3\cos^2 3\theta + \ldots + x\cos^2 x\theta = S_x$,

then $$S_x = \frac{x.(x+1)}{4} + \frac{x}{4}\,\frac{\sin(2x+1)\,\theta}{\sin\theta} - \frac{1}{4}\left(\frac{\sin x\theta}{\sin\theta}\right)^2.$$

9. If $\sin 2\theta\cos\phi + \sin 4\theta\cos 3\phi + \ldots + \sin 2x\theta\cos(2x-1)\,\phi = S_x$,

then $S_x = \frac{\sin x(\theta+\phi)}{2\sin(\theta+\phi)}\sin\{\theta + x(\theta+\phi)\} + \frac{\sin x(\theta-\phi)}{2\sin(\theta-\phi)}\sin\{\theta + x(\theta-\phi)\}$.

10. If $1^2\sec^2\theta + 2^2\sec^2 2\theta + 4^2\sec^2 4\theta + \ldots +$ to x terms, $= S_x$,

then $$S_x = 2^{2x}\operatorname{cosec}^2 2^x\theta - \operatorname{cosec}^2\theta.$$

11. Apply the expression in Art. 23 to sum the following series:

$$1^n - 2^n + 3^n - 4^n + \ldots\ldots \text{ to } \infty,$$

$$1^n + 2^n + 3^n + 4^n + \ldots\ldots \text{ to } \infty,$$

$$1^n + \frac{2^n h}{1} + \frac{3^n h^2}{1.2} + \frac{4^n h^3}{1.2.3} + \ldots \text{ to } \infty,$$

$$1^n - 3^n + 5^n - 7^n + \ldots\ldots \text{ to } \infty.$$

Hence shew that

$$1 - 1 + 1 - 1 \dots\dots \text{ to } \infty = \tfrac{1}{2}.$$

12. If $S_{2x+1} = \dfrac{1}{1^{2x+1}} - \dfrac{1}{3^{2x+1}} + \dfrac{1}{5^{2x+1}} - \dots\dots$ to ∞,

then $$S_{2x+1} = \frac{(-1)^x \left(\frac{\pi}{2}\right)^{2x+1}}{1.2..(2x)} \cdot \frac{1 + \Delta}{(1 + \Delta)^2 + 1} \cdot 0^{2x}.$$

Recurring Series.

1. Sum to x terms the following recurring series:

$$1 + 13 + 49 + 157 + \dots\dots$$

whose scale of relation is $1 + p + q$.

2. Sum also the following:

$$2 - a - a^2 + 2a^3 - a^4 - a^5 + 2a^6 - a^7 - a^8 + \dots\dots$$

3. The numbers $\Delta^n 0^n$, $\Delta^n 0^{n+1}$, $\Delta^n 0^{n+2} \dots$

form a recurring series whose scale of relation is

$$A_1 - A_2 + A_3 - \dots \pm A_n,$$

where $$A_1 = 1 + 2 + 3 + \dots + n,$$
$$A_2 = 1.2 + 2.3 + 1.3 + \dots + (n-1)\,n,$$
$$\dots\dots\dots\dots$$
$$A_n = 1.2.3 \dots n.$$

Interpolation.

1. Having given

$$L \tan 45^\circ = 0, \quad L \tan 45^\circ.30' = 0.0075803,$$
$$L \tan 46^\circ = 0.0151628, \quad L \tan 46^\circ.30' = 0.0227500;$$

find $$L \tan 45^\circ.20'.20''.$$

2. Given $\log_{10} 510 = 2.70757018,$

$$\log_{10} 511 = 2.70842090,$$

$$\log_{10} 513 = 2.71011737,$$

$$\log_{10} 514 = 2.71096312;$$

it is required to insert the deficient value

$$\log_{10} 512 = 2.70926996.$$

3. Given v_0, v_1, v_4, v_5; required v_2 and v_3,

$$v_2 = \frac{-3v_0 + 10v_1 + 5v_4 - 2v_5}{10},$$

$$v_3 = \frac{-2v_0 + 5v_1 + 10v_4 - 3v_5}{10}.$$

4. Given $\tan^{-1} 10 = 1.471127674,$

$$\tan^{-1} 11 = 1.480136439,$$

$$\tan^{-1} 13 = 1.494024435,$$

$$\tan^{-1} 15 = 1.504228163;$$

shew that $\tan^{-1} 11.63 = 1.485022707.$

5. Find the function which, when x assumes the values 2, 4, 6, 8, shall assume the values 9, 10, 11, 12.

Table of the numerical values of $\Delta^m 0^n$ for all values of m and n from 1 to 10.

	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7	Δ^8	Δ^9	Δ^{10}
0^1	1									
0^2	1	2								
0^3	1	6	6							
0^4	1	14	36	24						
0^5	1	30	150	240	120					
0^6	1	62	540	1560	1800	720				
0^7	1	126	1806	8400	16800	15120	5040			
0^8	1	254	5796	40824	126000	191520	141120	40320		
0^9	1	510	18150	186480	834120	1905120	2328480	1451520	362880	
0^{10}	1	1022	55980	818520	5103000	16435440	29635200	30240000	16329600	3628800

METCALFE AND PALMER, PRINTERS, CAMBRIDGE.